DTU Library

The geometry of generalized flat ribbons

Raffaelli, Matteo

Publication date:
2019

Document Version
Publisher's PDF, also known as Version of record

The geometry of generalized flat ribbons

Matteo Raffaelli

Kongens Lyngby 2018

Technical University of Denmark
Department of Applied Mathematics and Computer Science
Richard Petersens Plads, building 324,
2800 Kongens Lyngby, Denmark
Phone +45 4525 3031
compute@compute.dtu.dk
www.compute.dtu.dk

Summary (English)

This thesis is concerned with the Riemannian geometry of developable surfaces in Euclidean space, their generalization and application.

One of our main interests is to model surfaces by means of intrinsically flat ribbons. Naturally, we consider tangent (first-order) approximations, where each ribbon has the same distribution of tangent planes, i.e., the same normal field, as the surface along a given curve. Each ribbon is thus isometric to a planar ribbon constructed along the so-called Cartan development of the original surface curve. We show that the planar and the approximating (bent) ribbons are dual, rolling-related, constructions. In particular, the geodesic torsion and the normal curvature of the surface curve completely determine the rotational part of the rolling as well as the ruling angle function of the ribbons. On this ground, we present a rolling-based method for approximating surfaces via collections of flat ribbons, which we call ribbonizations. They are in some sense akin to the triangulations typically used in finite element methods and in computer-aided geometric design.

In higher dimensions, we study the local problem of approximating a hypersurface by means of a flat hyper-ribbon along a prescribed curve. We show that the well-known two-dimensional condition for the existence and uniqueness of the approximating ribbon naturally extends to this more general setting.

In higher codimension, we limit our analysis to the family of flat and ruled (or *developable*) submanifolds. More precisely, we solve the following problem: Given a smooth distribution $\mathscr{D}$ of m-dimensional planes along a smooth regular curve γ in $\mathbb{R}^{m+n}$, find all m-dimensional developable submanifold of $\mathbb{R}^{m+n}$ that pass through γ and whose tangent bundle along γ is precisely $\mathscr{D}$. In particular, we give sufficient conditions for the local well-posedness of the problem, together with a parametric description of the solution.

Summary (Danish)

Nærværende rapport handler om geometrien af udfoldelige flader i Euklidiske rum samt om generaliseringer og anvendelser af sådanne flader. Et hovedtema er modellering og approksimation af flader ved hjælp af flade bånd med Gauss krumning 0 – de benævnes her *Cartan ribbons*. I udgangspunktet betragter vi tangentielle approksimationer, hvor hvert bånd løbende langs en given kurve har sammenfald mellem båndets tangentplan og fladens tangentplan, således at de to normal-vektorfelter er sammenfaldende langs kurven. Ethvert Cartan ribbon er isometrisk med et plant bånd, som konstrueres langs en kurve, den såkaldte *Cartan-development*, i planen ud fra den givne kurve på fladen. Vi viser, hvordan de isometriske plane og approksimerende (bøjede) bånd kan frembringes og identificeres via en entydigt bestemt *rulning* af den givne flade på planen med den givne kurve som kontakt-kurve. Rotationsdelen af rulningen er bestemt af den geodætiske torsion og normal-krumningen af den givne kurve. Metoden benyttes til konstruktion af globale approksimationer af flader via samlinger af Cartan ribbons. Sådanne 'ribboniseringer' er beslægtede med de trianguleringer, der typisk anvendes i elementmetoder og i geometrisk (CAD) design. I højere dimensioner studerer vi dernæst det tilsvarende lokale problem, dvs. approksimation af en hyperflade ved hjælp af et fladt (krumningstensor 0) hyperbånd langs en foreskrevet kurve på hyperfladen. Vi viser, at det 'to-dimensionale' resultat om eksistens og entydighed af de approksimerende hyperbånd i 3D naturligt kan generaliseres til enhver højere dimension. I højere co-dimension (f.eks. 2D fla-

der i 4D) betragter vi endelig det tilsvarende generelle problem: Givet en distribution $\mathcal{D}$ af m-dimensionale underrum af tangentrummet langs en glat regulær kurve γ i $\mathbb{R}^{m+n}$; Find alle m-dimensionale frembragte (ruled) delmangfoldigheder langs γ i $\mathbb{R}^{m+n}$, som har tangentrums-distributionen $\mathcal{D}$. Specielt finder vi tilstrækkelige betingelser for dette problems lokale løsning og en tilsvarende parametrisk beskrivelse af løsningen under disse betingelser.

Preface

This thesis is submitted in partial fulfillment of the requirements for obtaining the Ph.D. degree. The work has been carried out at the Department of Applied Mathematics and Computer Science (DTU Compute) at the Technical University of Denmark under the supervision of Steen Markvorsen and Jakob Bohr.

The main subject of the work is submanifold geometry. A prerequisite for reading this thesis is some familiarity with local Riemannian geometry, roughly equivalent to what Lee covers in [18].

List of papers

The work, carried out from December 2015 to November 2018, has so far resulted in the following papers—listed in chronological order—on which the thesis is based:

(A) Matteo Raffaelli, Jakob Bohr, and Steen Markvorsen, *Sculpturing surfaces with Cartan ribbons*, Proceedings of Bridges 2016, Tessellations Publishing, 2016, pp. 457–460.

(B) Matteo Raffaelli, Jakob Bohr, and Steen Markvorsen, *Cartan ribbonization and a topological inspection*, Proc. A **474** (2018).

(C) Jakob Bohr, Steen Markvorsen, and Matteo Raffaelli, *Newson's challenge and the volume of certain convex hulls*, submitted.

(D) Irina Markina and Matteo Raffaelli, *Flat approximations of hypersurfaces along curves*, Manuscripta Math., to appear.

(E) Matteo Raffaelli, *The geometric Cauchy problem for developable submanifold*, in preparation.

Structure of the thesis

The thesis is made up of two parts. Part I consists of four chapters and is aimed at introducing and discussing the results contained in Part II, where Papers A–E are included.

Part I is organized as follows. In Chapter 1 we give a general introduction to the topics discussed in the thesis. Chapter 2 deals with some mathematical prerequisites. In Chapter 3 we give a summary of the main results of the papers. Finally, in Chapter 4 we discuss some open questions related to the results obtained here that will be the subject of future research. Part I ends with the bibliography of the first four chapters.

Acknowledgements

First of all I want to thank my advisor Steen Markvorsen and my co-advisor Jakob Bohr for their guidance and constant support throughout the whole project. I am grateful for their encouragement to pursue my own ideas and interests, as well as for their availability to help whenever I have felt stuck with a problem.

Besides my advisors, I sincerely thank my co-author Irina Markina for the precious collaboration on our joint project, and for her warm hospitality during the four months spent at the University of Bergen, which I really enjoyed. I also thank David Brander for helpful feedback on Paper E and for putting me in contact with Ruy Tojeiro.

A big thank goes to everyone at the Section for Mathematics for making these last three years so enjoyable. I believe working in a stimulating and peaceful environment is a necessary condition for doing anything good in mathematics.

Last but not least, I thank my family and my girlfriend, as I would have never reached the end of the Ph.D. without their moral support.

Contents

Part I

Outline of the research

CHAPTER 1

Introduction

A developable surface is one of the classical geometrical objects that many scientist have studied with great interest in the past. Their history can be traced as far as back as Aristotle, although it is only with the discovery of differential calculus that they started to be studied in greater depth [17]. Arguably, it is because of the rich collection of properties they possess that they became so attractive. Indeed, a developable surface [24, p. 388]

1. is locally isometric to the plane, i.e., can be locally flattened, or "developed" onto the plane without any stretching or compressing;
2. has zero Gauss curvature;
3. is an envelope of a one-parameter family of planes;
4. is a so-called *torse*, that is, a ruled surface with tangent plane stable along the rectilinear rulings.

Although one typically chooses the first as defining property, it is remarkable that any of 1–4 qualifies as characteristic.

For centuries cylinders and cones were believed to be the only surfaces of this family. It was only in the eighteenth century, in particular with the work of Euler and Monge, that the classification task was completed by including tangent surfaces. Hence, *locally* there are only three types of non-planar developable surfaces—see for example [8, Section 3-5]:

(i) Generalized cylinders, where all the generating lines, or rulings, are parallel;

(ii) Generalized cones, where the rulings pass through a fix point;

(iii) Tangent surfaces, where the rulings are spanned by the tangent vector of some curve, the so-called *edge of regression* of the developable.

Not only have developable surfaces a rich and fascinating history. They also have a number of important modern applications in diverse fields such as cartography, architecture, manufacturing, to mention but a few—see references in Paper B.

In differential geometry, developable surfaces are the prototypical examples of flat Riemannian (sub)manifolds. As such, they continue to inspire geometers and to be of great help in understanding more general families of manifolds.

1.1 D-forms

D-forms provide an interesting recent application of developable surfaces to art and mathematics. They were invented by the artist Tony Wills [25] and introduced in the literature in [20].

To construct a D-form, take two sheets of paper bounded by closed convex curves c_1 and c_2 of equal length. Choose two points $p_1 \in c_1$ and $p_2 \in c_2$. Finally, bend the two sheets in space so that both the curves and the selected points come to coincidence.

Mathematically speaking, we may define a *D-form* as a convex piecewise-C^2 flat surface S with precisely two flat components (maximal connected flat open sets), each isometric to a convex plane region with smooth (C^∞) boundary. Points of S belonging to both (closures of the) flat components form the so-called *seam curve.*

Rather surprisingly, it turns out that the existence of *creases*—i.e., C^2 edges where the surface itself fails to be C^1—is only possible along the seam curve. In addition, every D-form is the convex hull of its seam, see [7].

CHAPTER 2

Background

2.1 Frames along curves

In this section we define frames along curves in a manifold, and then consider particular examples of these. Since we are only interested in smooth objects (manifolds, maps, etc.), we typically omit the word "smooth".

Let $M = M^m$ be an m-dimensional manifold, and let γ be a regular curve in M, that is, an immersion $I \to M$, where I is an interval. In particular, regularity means that $\dot{\gamma}(t) \neq 0$ for all $t \in I$.

Remember that a *vector field along* γ is a map W from I to the tangent bundle TM of M such that $W(t) \in T_{\gamma(t)}M$ for every $t \in I$. We define a *frame along* γ to be an m-tuple $(E_1, \ldots, E_m)$ of vector fields along γ such that the vectors $(E_i(t))_{i=1}^m$ form a basis of $T_{\gamma(t)}M$ for every $t \in I$.

We further say that a frame along γ is γ-adapted whenever E_1 coincides with the tangent vector $\dot{\gamma}$.

Let now $g = \langle \cdot, \cdot \rangle$ be a Riemannian metric on M. The presence of the metric allows us to talk about lengths and angles in each tangent space. Hence, we say that a frame $(E_1, \ldots, E_m)$ along γ is *orthonormal* if $g(E_i, E_l) = \delta_{il}$ for all $i, l \in \{1, \ldots, m\}$.

Assume $(E_i)_{i=1}^m$ be a γ-adapted orthonormal frame along the (unit-speed) curve γ. Denoting by D_t the covariant derivative along γ induced by the Levi-Civita connection of (M, g), we may express the covariant derivative $D_t E_i$ of E_i with respect to the frame itself, as follows:

$$D_t E_i = \langle D_t E_i, E_1 \rangle E_1 + \cdots + \langle D_t E_i, E_m \rangle E_m \, .$$

Note, in particular, that $\langle D_t E_i, E_i \rangle = 0$ and $\langle D_t E_i, E_l \rangle = -\langle E_i, D_t E_l \rangle$.

2.1.1 Classical examples

2.1.1.1 The Frenet–Serret frame

Let γ be a unit-speed curve in $\mathbb{R}^3$. Under the additional non-trivial assumption that $\ddot{\gamma}(t) \neq 0$, we define the *curvature* $\kappa_n(t)$ *of* γ *at* t to be the Euclidean length of $\ddot{\gamma}(t)$:

$$\kappa(t) = \sqrt{\ddot{\gamma}(t) \cdot \ddot{\gamma}(t)}. \tag{2.1}$$

Since $\dot{\gamma} \cdot \dot{\gamma} = 1$, it follows that $\ddot{\gamma} \cdot \dot{\gamma} = 0$, and so the unit vector $\ddot{\gamma}(t)/\kappa(t)$ is in the normal space of γ at t (the orthogonal complement in $\mathbb{R}^3$ of $\dot{\gamma}(t)$). Letting $v_1 = \dot{\gamma}(t)$ and $v_2 = \ddot{\gamma}(t)/\kappa(t)$, we define v_3 to be the cross product $v_1 \times v_2$, and we call the quantity $\dot{v}_2 \cdot v_3 = \tau(t)$ the *torsion* of γ at t.

Clearly, the vectors v_1, v_2, v_3 are pairwise orthogonal and have unit length. In short, they are *orthonormal.* If the condition $\kappa(t) \neq 0$ holds for all

$t \in I$, then we may extend this construction to the entire interval. The resulting triple of vector fields (V_1, V_2, V_3) along γ is called the *Frenet–Serret frame* of γ.

It is easy to see that the following relations hold:

$$\dot{V}_1 = \kappa V_2 \,, \tag{2.2}$$

$$\dot{V}_2 = -\kappa V_1 + \tau V_3 \,, \tag{2.3}$$

$$\dot{V}_3 = -\tau V_2 \,. \tag{2.4}$$

The functions κ and τ have important geometrical significance. In particular, they uniquely define a given curve in $\mathbb{R}^3$ up to a rigid motion of the ambient space.

2.1.1.2 The Darboux frame

Now suppose the unit-speed curve $\gamma \colon I \to \mathbb{R}^3$ lies on a surface $S^2 \subset \mathbb{R}^3$, that is, $\gamma(t) \in S$ for all $t \in I$. If the surface in question is orientable, then we may choose a unit normal vector field N along S. Identifying N with $N \circ \gamma$, we set $W_1 = \dot{\gamma}$, $W_2 = N \times \dot{\gamma}$, and $W_3 = N$.

The orthonormal frame $(W_i)_{i=1}^3$ is a natural choice of frame along γ, for it is both γ-adapted and S-adapted—i.e., not only $W_1(t) = \dot{\gamma}(t)$, but the vectors $W_1(t)$ and $W_2(t)$ span the tangent plane of S at $\gamma(t)$. The frame $(W_i)_{i=1}^3$ is classically known as the *Darboux frame* of γ on S.

This construction easily extends to the case where the surface S is not orientable. Indeed, instead of basing the definition on a normal vector field along S, we could have used any (smooth) section of the normal bundle of S along γ.

In any case, the acceleration vector $\ddot{\gamma}(t)$ is orthogonal to the unit vector $\dot{\gamma}(t)$ for every t. It follows that there exist smooth functions κ_n, κ_g, called the *normal curvature* and the *geodesic curvature* of γ, respectively, such that:

$$\ddot{\gamma} = \kappa_n N + \kappa_g (N \times \dot{\gamma}) \,.$$

Similarly, letting $\tau_g = \dot{W}_2 \cdot N$—the so-called *geodesic torsion* of γ—we have the following relations expressing each derivative in terms of the frame itself:

$$\dot{W}_1 = \kappa_g W_2 + \kappa_n W_3 \,, \tag{2.5}$$

$$\dot{W}_2 = -\kappa_g W_1 + \tau_g W_3 \,, \tag{2.6}$$

$$\dot{W}_3 = -\kappa_n W_1 - \tau_g W_2 \,. \tag{2.7}$$

2.1.2 The Frenet frame

The most well-known example of an orthonormal frame along a curve is surely the classical Frenet–Serret frame of a curve in three-dimensional Euclidean space. Following [23], we shall present how such construction generalizes when the ambient space is an arbitrary Riemannian manifold. To this end, it is necessary to limit our attention to a specific subclass of curves in M^m, those having the property of being C^m-regular (C^{m-1}-regular if M^m is oriented).

For any integer $l \in \{1, \ldots, m\}$, we say that γ is C^l*-regular* if the first $l-1$ covariant derivatives of $\dot{\gamma}$ are linearly independent. In other words, we require the vectors

$$\dot{\gamma}(t), D_t\dot{\gamma}(t), \ldots, D_t^{(l-1)}\dot{\gamma}(t)$$

be linearly independent for every t.

Now, if γ is a C^l-regular curve in M, we may define $l-1$ smooth functions $\kappa_1, \ldots, \kappa_{l-1}$ and l smooth vector fields $V_1, \ldots, V_l$ along γ as follows. Let $V_1 = \dot{\gamma}$ and $V_2 = D_t V_1/\kappa_1$, where $\kappa_1 = |D_t V_1|$. Then, for $2 \leq s \leq l-1$, let

$$\begin{aligned} \kappa_s &= |D_t V_s + \kappa_{s-1} V_{s-1}| \,, \\ V_{s+1} &= (D_t V_s + \kappa_{s-1} V_{s-1})/\kappa_s \,. \end{aligned}$$

It is not difficult to see that the vector fields $V_1, \ldots, V_l$ are well defined. If γ is C^m-regular, then the m-tuple $(V_1, \ldots, V_m)$ is a γ-adapted orthonormal frame along γ.

Provided γ is a C^m-regular curve in M, we call the m-tuple $(V_1, \dots, V_m)$ the *Frenet frame* of γ. The (strictly positive) function κ_j $(1 \leq j \leq m-1)$ and the vector field V_l $(1 \leq l \leq m)$ are called the j-th *geodesic curvature* and the l-th *Frenet vector* of γ, respectively. The Frenet vectors satisfy the following equations:

$$\begin{aligned}
D_t V_1 &= \kappa_1 V_2 \,, \\
D_t V_2 &= \kappa_2 V_3 - \kappa_1 V_1 \,, \\
&\ \vdots \\
D_t V_j &= \kappa_j V_{j+1} - \kappa_{j-1} V_{j-1} \,, \\
&\ \vdots \\
D_t V_{m-1} &= \kappa_{m-1} V_m - \kappa_{m-2} V_{m-2} \,, \\
D_t V_m &= -\kappa_{m-1} V_{m-1} \,.
\end{aligned} \tag{2.8}$$

If M carries an orientation, the extra structure allows for the introduction of a somewhat improved version of the Frenet frame. Keeping the same definitions as before for the first $m-1$ Frenet vectors and the first $m-2$ geodesic curvatures, we let

$$V_m = V_1 \times \cdots \times V_{m-1} \,, \tag{2.9}$$

$$\kappa_{m-1} = -\langle V_{m-1}, D_t V_m \rangle \,, \tag{2.10}$$

where "$\times \cdots \times$" denotes the positively oriented $(m-1)$-fold vector cross product on M, see Paper D.

Clearly, since we no longer need to divide by κ_{m-1}, this new construction has the advantage of being well defined on a larger class of curves in M, requiring one less degree of regularity. For instance, in dimension two it is defined for *any* curve in M (including geodesics!). Without further comment, we adopt the new definitions of V_m and κ_{m-1} whenever possible.

Example 2.1 (The Frenet–Serret frame)**.** Let $M = \mathbb{R}^3$, with the usual Euclidean metric $\overline{g}$ and standard orientation, and assume γ is a C^2-regular curve, i.e., that $\overline{D}_t \dot{\gamma}(t) \neq 0$ for every t. In this situation, the newly defined frame is nothing other than the classical Frenet–Serret frame of γ. The first geodesic curvature $\kappa_1 = |\overline{D}_t \dot{\gamma}(t)| > 0$ is the curvature κ of γ, whereas the real valued function $\kappa_2 = \overline{g}(v_3, \overline{D}_t v_2)$ its torsion τ.

Example 2.2 (Curves on a surface)**.** Let M be a two-dimensional embedded submanifold of $(\mathbb{R}^3, \overline{g})$, with the induced metric. Assuming M is orientable, we give M the orientation induced by a unit normal vector field N along M and by the standard orientation on $\mathbb{R}^3$. For *any* curve γ in M, the Frenet frame of γ is $(\dot\gamma, H \times \dot\gamma)$, where $H = N \circ \gamma$ and "$\times$" denotes the ordinary cross product on $\mathbb{R}^3$. There is only one geodesic curvature,

$$\kappa_1 = \overline{g}(D_t\dot\gamma, H \times \dot\gamma)\,.$$

Since the Riemannian connection of M coincides with the tangential connection, $D_t\dot\gamma(t)$ is simply the orthogonal projection on the tangent space $T_{\gamma(t)}M$ of the $\overline{g}$-acceleration $\overline{D}_t\dot\gamma(t)$ of γ. In short, κ_1 is just the "usual" geodesic curvature κ_g.

Remark 2.3. In the literature, the quantity $|D_t\dot\gamma|$ is usually referred to as *the* geodesic curvature of γ. As a matter of fact, $|D_t\dot\gamma|$ is important in that it gives a measure of how much γ deviates from being a geodesic (for which $|D_t\dot\gamma| = 0$).

2.1.3 Curves on hypersurfaces

In this subsection we consider the case when M^m is an embedded hypersurface of a Riemannian manifold $(\widetilde{M}^{m+1}, \widetilde{g})$.

In this setting, under the canonical identification of the tangent space T_pM with its image under the differential of the inclusion $\iota\colon M \hookrightarrow \widetilde{M}$, the ambient tangent space $T_p\widetilde{M}$ decomposes as the orthogonal direct sum $T_pM \oplus N_pM$, and any vector field along $\gamma\colon I \to M$ may be naturally thought of as a *tangent* vector field along $\widetilde{\gamma} = \iota \circ \gamma$.

We now equip M with the induced metric $\iota^*\widetilde{g}$, and we assume that the ambient manifold carries an orientation. If γ is a curve in M with a well defined Frenet frame $(V_1, \dots, V_m)$, we construct an orthonormal frame $(W_1, \dots, W_{m+1})$ along $\widetilde{\gamma}$ as follows. For any $l \in \{1, \dots, m\}$, we require $W_l = V_l$. Then we let

$$W_{m+1} = V_1 \times \cdots \times V_m\,.$$

We call $(W_1, \ldots, W_{m+1})$ the *Darboux frame* of $\widetilde{\gamma}$. It is the unique extension of the Frenet frame of γ to a positively oriented, orthonormal frame along $\widetilde{\gamma}$.

Because W_{m+1} spans the normal bundle of M along γ, the ambient covariant derivative $\widetilde{D}_t V_l$ of the l-th Frenet vector has decomposition

$$\widetilde{D}_t V_l = D_t V_l + \tau_l W_{m+1} \,,$$

for some smooth function $\tau_l \colon I \to \mathbb{R}$. Note that, if M is orientable, then $\tau_l = h(V_1, V_l)$, where h is the scalar second fundamental form of M determined by the unit normal vector field extension of W_{m+1}. It is straightforward to verify that

$$\widetilde{D}_t W_{m+1} = -\tau_1 V_1 - \cdots - \tau_m V_m \,.$$

Example 2.4 (The classical Darboux frame)**.** In the case of example 2.2, the function $h(V_1, V_1)$, which is the normal component of the acceleration of $\widetilde{\gamma}$, is nothing but the normal curvature κ_n of γ, whereas $h(V_1, V_2)$ equals the geodesic torsion τ_g of $\widetilde{\gamma}$.

2.2 Flat approximations of surfaces along curves

Given a curve $\gamma \colon I \to S$ in some smooth surface $S \subset \mathbb{R}^3$, we consider the problem of constructing a *developable* surface which passes through γ and has the same tangent plane as S at all points of the curve. We call any such surface a *flat approximation of S along γ*.

Using the fact that flat surfaces without planar points are the same as ruled surfaces with constant normal along the rulings, we start by defining a parametrization $\sigma \colon I \times U \to \mathbb{R}^3$ as follows:

$$\sigma(t, u) = \gamma(t) + uX(t). \tag{2.11}$$

Here U is an interval containing 0, and $X \colon I \to \mathbb{R}^3$ a non-vanishing vector field along γ. In particular, for regularity purposes, we shall assume U be sufficiently small and $X(t)$ never parallel to the tangent vector $\dot{\gamma}(t)$.

Computing the partial derivatives of σ, we obtain:

$$\begin{aligned}\frac{\partial\sigma}{\partial t}(t,u) &= \dot{\gamma}(t) + u\dot{X}(t),\\ \frac{\partial\sigma}{\partial u}(t,u) &= X(t),\end{aligned}$$

Their cross product defines a normal vector field $Z\colon I \times U \to \mathbb{R}^3$ along σ:

$$Z(t,u) = \dot{\gamma}(t) \times X(t) + u\dot{X}(t) \times X(t).$$

Clearly, the normal space of σ—considered as a subspace of $\mathbb{R}^3$—is constant along the ruling $L(t) \subset \operatorname{span} X(t)$ if and only if the vector field Z is parallel along $L(t)$. Hence, σ is a torse precisely when

$$\pi^{\top}\frac{\partial Z}{\partial u} = 0, \tag{2.12}$$

where $\pi^{\top}$ denotes the orthogonal projection onto the tangent space of σ.

Moreover, from

$$\frac{\partial Z}{\partial u}(t,u) = \dot{X}(t) \times X(t), \tag{2.13}$$

we observe that $\frac{\partial Z}{\partial u} \cdot \frac{\partial \sigma}{\partial u} = 0$. It follows that (2.12) is equivalent to

$$\frac{\partial Z}{\partial u} \cdot \frac{\partial \sigma}{\partial t} = (X \times \dot{\gamma}) \cdot \dot{X} = 0.$$

Thus, we may conclude that:

Theorem 2.5. *The map σ parametrizes a torse surface if and only if*

$$\dot{N} \cdot X = 0, \tag{2.14}$$

where $N\colon I \to \mathbb{R}^3$ is a non-vanishing vector field—always normal to σ—along γ.

Proof. It is sufficient to note that $X(t) \times \dot{\gamma}(t) = Z(t,0)$ and $Z(t,0) \cdot \dot{X}(t) = -\frac{\partial Z}{\partial t}(t,0) \cdot X(t)$. $\square$

In view of this result, we formulate the flat approximation problem as follows:

Problem 2.6. Let γ be a curve in a smooth surface $S^2 \subset \mathbb{R}^3$. Let N be a *unit normal* vector field along γ. Find a vector field X along γ such that, for all $t \in I$,

$$\begin{aligned} &X(t) \notin \operatorname{span} \dot{\gamma}(t), \\ &X(t) \in N(t)^{\perp} \cap \dot{N}(t)^{\perp} = T_{\gamma(t)}S \cap \dot{N}(t)^{\perp}, \end{aligned} \tag{2.15}$$

where the superscript $^{\perp}$ denotes orthogonal complement in $\mathbb{R}^3$.

The solution is presented in the following theorem—see [8, p 195–197] and [15].

Theorem 2.7. *Assume γ is never parallel to an asymptotic direction of S, i.e., the normal curvature $\kappa_n = \ddot{\gamma} \cdot N$ of γ never vanishes. Then there exists a vector field satisfying* (2.15)*. Moreover:*

1. *Such vector field is* locally unique*: if X_1 and X_2 are both solutions of* (2.15)*, then* $\operatorname{span} X_2 = \operatorname{span} X_1$*;*
2. *The locally unique solution X is given by*

$$X = \kappa_n V_2 - \tau_g V_1\,, \tag{2.16}$$

where (V_1, V_2) is the Frenet frame of $\gamma \colon I \to S$.

Proof. Suppose $\dot{\gamma}(t) \cdot \dot{N}(t) = -\kappa_n(t) \neq 0$. Then $\dot{\gamma}(t) \notin \dot{N}(t)^{\perp}$ and the intersection $T_{\gamma(t)}S \cap \dot{N}(t)^{\perp}$ has dimension one, as desired. Since $\dot{N}(t) \times N(t) \in T_{\gamma(t)}S \cap \dot{N}(t)^{\perp}$, we compute

$$\begin{aligned} \dot{N}(t) \times N(t) &= -\left(\kappa_n(t)V_1(t) + \tau_g(t)V_2(t)\right) \times N(t) \\ &= \kappa_n(t)N(t) \times V_1(t) - \tau_g(t)V_2 \times N(t) \\ &= \kappa_n(t)V_2(t) - \tau_g(t)V_1(t)\,. \end{aligned}$$

□

2.3 Some generalizations of developable surfaces

In this section we give a general description of certain classes of Euclidean submanifolds which extends the notion of a developable surface. We refer the reader to Rovenskii's survey [22], and references therein, for further generalizations to non-Euclidean ambient spaces.

2.3.1 Developable hypersurfaces

Developable hypersurfaces in $\mathbb{R}^{m+1}$ enjoy essentially the same set of (characteristic) properties of the flat surfaces in $\mathbb{R}^3$. More precisely [24, p. 389], they

1. are locally isometric to $(\mathbb{R}^m, \overline{g})$, where $\overline{g}$ is the standard Euclidean metric;

2. have vanishing sectional curvatures;

3. are envelopes of one-parameter families of hyperplanes;

4. are higher-dimensional torses, i.e., they are foliated by (open subsets of) $(m-1)$-dimensional affine subspaces of $\mathbb{R}^{m+1}$, along which the tangent space remains constant.

Some important results concerning developable hypersurfaces are contained in [14, 3]. In particular, a developable hypersurface free of planar points can be locally represented in terms of its Gauss map (a curve on the sphere) and its support function, see [4, 10].

In fact, if one regards two hypersurfaces being distinct even when related by a rigid motion of the ambient Euclidean space, the Gauss parametrization—which maps a flat hypersurface to the pair determined by its Gauss image and its support function—gives a bijection onto the product $\{\text{curves on the sphere}\} \times \{\text{smooth functions on the interval}\}$.

2.3.2 Developable submanifolds

Developable submanifolds of Euclidean space form the class of submanifolds most fully inheriting the rich properties of the classical developable surfaces in $\mathbb{R}^3$. Indeed, a submanifold is *developable* if

1. it is *ruled*—i.e., it is free of totally geodesic points and is foliated by open subsets of $(m-1)$-dimensional affine subspaces of $\mathbb{R}^{m+n}$—*and*

2. with the natural metric induced from the standard Euclidean metric, it is a flat Riemannian manifold.

Note that for a ruled submanifold the flatness of the metric is equivalent to the constancy of the tangent plane along the rulings, see Paper E.

Moreover, developable submanifolds may be characterized in terms of their first normal space (span of the image of the second fundamental form):

Theorem 2.8 ([24, Theorem 3])**.** *A smooth submanifold of* $\mathbb{R}^{m+n}$ *is developable precisely when its induced metric is flat and the dimension of its first normal space is one everywhere.*

It thus follows from Erbacher theorem [9] that *full* developable submanifolds (those not contained in any affine plane of dimension smaller than $m+n$) have nonparallel first normal bundle, see e.g. [1, Chapter 1] for more details. Submanifolds in spaces of constant curvature with one-dimensional first normal bundle are studied in [13]; those with nonparallel first normal bundle in [5, 6].

We should finally emphasize that, in codimension higher than one, being flat does not imply being ruled. Indeed, the Clifford torus, the product of two circles in $\mathbb{R}^4$, is the standard example of a submanifold which is flat but does not possess a ruled structure.

2.3.3 Submanifolds of constant nullity

These are the m-dimensional submanifolds $M^m \subset \mathbb{R}^{m+n}$ which are foliated by open subset of k-dimensional affine subspaces of $\mathbb{R}^{m+n}$, along which the tangent space is constant (up to parallel translation in $\mathbb{R}^{m+n}$, of course). Hence, they generalize developable submanifolds in that they have rulings of arbitrary dimension.

Equivalently, they may be defined as the submanifolds whose index of relative nullity is a positive constant. Such quantity, which was introduced by Chern and Kuiper in [2], is the dimension of the nullity space of the second fundamental form.

These submanifolds have been extensively studied during the sixties. A proof of their widespread popularity comes from the fact they have been constantly renamed, see [24, p. 390]. Unfortunately, not always in a completely meaningful way. Indeed, some authors use the term "k-developable surfaces", although in fact they are generally non flat, and so they cannot be "developed" anywhere—at least in the classical (planar) sense.

Chapter 3

Main results

3.1 Project 1: Cartan ribbonization of surfaces

3.1.1 Motivation

As already suggested by their name, developable surfaces admit local isometric bending onto the plane. One of the main purposes of Paper A and B is to illustrate the intimate relation between developable surfaces and the classical notion of rolling of a manifold on another. At the same time, we wanted to study the problem of approximating a surface in space by means of developable ribbons, as an alternative approach to more classical methods, e.g. that of triangulations.

The starting point of the study was a seminal paper by K. Nomizu [19], where he presents a series of geometric conditions for the existence of a rolling (without skidding or twisting) of a given surface S on its tangent plane along a given curve γ. He shows that, provided the normal curvature and the geodesic torsion of γ are never simultaneously zero, then such rolling exists uniquely. Moreover, the image of γ under the motion is precisely the *Cartan development* [16, p. 131] of the original curve.

3.1.2 Summary: Paper A

In this work we introduce a rolling-based method for constructing, or "sculpting" a given surface S in space by means of planar, strip-shaped surfaces, which we call Cartan ribbons.

A planar Cartan ribbon can be bent (without stretching or compressing) to approximate a surface S along a curve γ. We employ the rolling of S on its tangent plane $T_{\gamma(0)}S$ as a tool for understanding the isometry between the planar and the bent ribbons.

In particular, if the normal curvature of γ is non-zero, then a flat approximation of S along γ exists and is parametrized by

$$\sigma(t,u) = \gamma(t) + uX(t)\,,$$

where X is given by (2.16).

The planar isometric image of σ is thus

$$\widehat{\sigma}(t,u) = \widehat{\gamma}(t) + u\widehat{X}(t)\,,$$

where $\widehat{\gamma}$ is the Cartan development of γ and $\widehat{X}(t)$ the parallel transport of $X(t)$ from $\gamma(t)$ to $\gamma(0)$ along γ.

The rolling gives a measure of how much bending one needs to apply—along the direction defined by the vector field $\widehat{X}$ along $\widehat{\gamma}$—to obtain the desired flat approximation of S along γ.

3.1.3 Summary: Paper B

In this article we give a more detailed description of the concepts introduced in Paper A, and present a topological application.

In particular, we introduce a measure for the local goodness of a single ribbon approximation. Such measure is just the volume enclosed by the ribbon and its normal shadow onto the surface.

Moreover, we show that two neighboring ribbons can be always extended until one intersects the other, provided their center curves are sufficiently close. This result may be considered as a first step towards a proof of existence of a global ribbonization of any given surface.

Concerning the global structure of the surface, we present a simple method for determining the Euler characteristics from any sufficiently fine Cartan ribbonization, which amounts to counting the degree of each of its vertices. This approach represents an alternative to the classical methods based on Morse theory and Poincaré–Hopf index theorem.

3.2 Project 2: Volume of the convex hull of closed curves

3.2.1 Motivation

It is quite surprising that some intuitive problems concerning the volume of the convex hull of a space curve are still without resolution, see for example [12] and references therein. A notorious one is the following isoperimetric-type problem:

Problem 3.1 ([12, Problem 5.2])**.** Let γ be a closed curve of fixed length L in $\mathbb{R}^3$. How big can the volume of the convex hull of γ be?

In general, the convex hull of a closed space curve γ contains both planar faces as well as developable surfaces. However, if each of the supporting planes touches γ in exactly two points, then the hull is free of planar regions [20, p. 400–401].

The result below—which follows easily from generalized versions of the four-vertex theorem, see e.g. [21, Corollary 1]—characterizes simple D-forms as convex hulls of their seam curves:

Proposition 3.2. *Assume that γ is convex, i.e., it lies on the boundary of its convex hull. If γ has precisely four points of zero torsion and no point of zero curvature, then γ has no tri-tangential supporting planes and the convex hull of γ is a simple D-form without planar regions.*

Proof. From [21, Corollary 1], we have that $V + 2K + 2d \geq 4 + P$, where V is the number of vertices (points of zero torsion), K the number of zero curvature points, d the number of external segments, and P the number of support polygons. Note that d is by definition zero if γ is convex. $\square$

3.2.2 Summary: Paper C

We obtain a surprisingly simple formula for the volume of the convex hull of any space curve γ satisfying the following requirements: γ is convex, has four vertices and non-vanishing curvature. In light of Proposition 3.2, our formula yields the volume of any simple D-form. The general idea for the proof involves parametrizing the convex hull and observing that the chosen map covers all its internal points exactly four times. Changes in orientation are handled by taking the absolute value of the volume element. The requirement that the internal points of the hull are covered an equal number of times by the parameterization is what limits the validity of the formula to closed space curves with only 4 vertex points.

3.3 Project 3: Cauchy problems for submanifolds with nullity

3.3.1 Motivation

Given a smooth regular curve on a smooth surface S in $\mathbb{R}^3$, we have discussed in Section 2.2 the problem of constructing a flat approximation of S along γ. It is clear that the solution, when defined, depends on the original surface S only through its distribution of tangent planes along

γ. We may thus consider the flat approximation problem as a particular example of a more general question.

Given a smooth $(m+n)$-manifold Q^{m+n} and some class $\mathcal{A}^m$ of m-dimensional embedded submanifolds of Q^{m+n}, we can formulate the *geometric Cauchy problem* for the class $\mathcal{A}^m$ as follows:

Problem 3.3. Let $\gamma \colon I \to Q^{m+n}$ be a smooth regular curve in Q^{m+n}, and let $\mathscr{D}$ denote a smooth distribution of rank m along γ, such that $\dot{\gamma}(t) \in \mathscr{D}_t$ for all $t \in I$. Find all members of $\mathcal{A}^m$ containing γ and whose tangent bundle along γ is precisely $\mathscr{D}$.

Remark 3.4. In case Q^{m+n} is a Riemannian manifold, let $\mathscr{D}_t^\perp$ be the orthogonal complement of $\mathscr{D}$ in the tangent space $T_{\gamma(t)}Q^{m+n}$. Then, problem 3.3 is of course equivalent to finding all members of $\mathcal{A}^m$ containing γ and whose *normal* bundle along γ coincides with the *orthogonal distribution* $\mathscr{D}^\perp = \bigcup_{t\in I} \mathscr{D}^\perp$.

We have seen that Problem 3.3—when addressed to the class of flat surface in $\mathbb{R}^3$—is generally well-posed:

Theorem 3.5. *Let N be a vector field—always normal to S—along γ. Assume the function $\gamma'' \cdot N$ is never vanishing, i.e., that the curve is never parallel to an asymptotic direction of S. Then the geometric Cauchy problem for developable surfaces in $\mathbb{R}^3$ has a solution, and such solution is locally unique.*

It is thus natural to look for extensions of this result to more general submanifolds.

3.3.2 Summary: Paper D

In this paper—still using the language of approximations—we examine the case of developable hypersurfaces.

The main result of the paper is that Theorem 3.5 naturally extends to higher dimensions. In other words, the sole condition for the existence of the desired approximation is that γ is never parallel to an asymptotic direction of M. In particular, using the characterization of flat surfaces without planar points as torses (ruled surfaces with tangent plane stable along the rulings), we prove that there is a unique local solution.

The proof is based on the fact that a suitable restriction of the map $\gamma + \mathrm{span}(X_j)_{j=1}^{m-1}$ is a parametrized torse surface, possibly with planar points, if and only if the Euclidean covariant derivatives $\overline{D}_t X_1, \dots, \overline{D}_t X_{m-1}$ along γ have vanishing normal component. Here γ is a curve in M, while $X_1, \dots, X_{m-1}$ are linearly independent vector fields along γ, such that $\dot{\gamma} \notin \mathrm{span}(X_j)_{j=1}^{m-1}$.

More specifically, this requirement on the covariant derivatives determines a system of $m-1$ equations involving $(X_j)_{j=1}^{m-1}$. The presented solution method requires to choose a smooth orthonormal frame for the normal bundle $N\gamma \subset TM$ of γ. In order to get rid of the ambiguity between tuples of vector fields and distributions, we use the generalized vector cross product.

3.3.3 Summary: Paper E

The main objective of this paper is to further generalize Theorem 3.5 to the class of developable submanifolds of $\mathbb{R}^{m+n}$, while at the same time simplifying the proof of the main theorem in Paper D.

We show that, with the increase in codimension, an additional non-trivial condition for the existence of the solution appears. Such condition guarantees that the first normal space of the solution has dimension one at every point.

We prove this result in two different ways. First, by a constructive approach based on Grassmannians, which turns out to be much more direct than the argument used in Paper D. Then we also include a coordinate-free method, which is arguably more elegant, but requires extra efforts for constructing the solution.

This paper may be considered an improvement of Paper D also because it formulates the question as an actual geometric Cauchy problem, thus simplifying the discussion quite a lot.

Chapter 4

Conclusions and future work

4.1 Cartan ribbonization of surfaces

The work presented in Paper A and B is local in nature, although Paper B deals with some global questions as well. However, the problem of existence of a global Cartan ribbonization of any given surface remains open.

Another intriguing question is concerned with the choice of center curves. What is the minimal set of center curves necessary to capture the topology of the underlying surface? If one is merely interested in finding the minimum number of center curves, we believe that the answer is one.

However, a more meaningful interpretation of minimality should probably take into account the overall length of the center curves. In this case, we expect the answer to be depending on the topology itself.

4.2 Volume of the convex hull of closed curves

The formula presented in Paper C for the volume of the convex hull of a space curve works under rather strong assumptions. Indeed, its applicability is confined to convex curves with precisely four points of zero torsion and no point of zero curvature.

The natural continuation of the work would be to extend the validity of the formula to larger classes of curves in $\mathbb{R}^3$. For example, space curves with more than four vertices. It would also be interesting to look for a higher-dimensional analogue of our result.

4.3 Cauchy problems for submanifolds with nullity

Having solved the Cauchy problem for flat ruled submanifolds in any codimension, it would be interesting to consider the Cauchy problem for flat submanifolds. Indeed, it is likely that the *ruled* solution described in Paper E is not the unique flat solution, i.e., one should expect many more (non-ruled) flat solutions.

In this regard, whilst the uniqueness—i.e., the isometric rigidity—of the ruled solution seems out of question, its isometric flexibility is far less obvious, at least for relatively low codimensions. In other words, one would like to understand the moduli space (the space of deformations) of the flat metric defined on the class of submanifolds with tangent spaces in the distribution $\mathscr{D}$ along γ.

The next step could be to study the case where $n < m$. As pointed out by Ruy Tojeiro (private communication), it is classically known that every flat m-submanifold of codimension $n < m$ has a constant index of relative nullity (dimension of the kernel of the second fundamental form) at least equal to $m-n$. This observation suggests to address the following question: among all submanifolds of constant rank, which are the flat ones? For instance, in codimension 2, submanifolds of rank 2 divide into three classes, hyperbolic, parabolic or elliptic, according to the number 0, 1 or 2 of independent normal directions whose shape operators have rank one. Flat submanifolds of rank 2 are the hyperbolic submanifolds for which such normal directions are everywhere orthogonal, see [11].

Alternatively, a more direct approach could be to consider a normal variation of the ruled solution preserving the tangent bundle along γ, and study the system of partial differential equations one obtains from the zero curvature requirement.

The conjecture on the existence of non-ruled flat solutions, even for $n < m$, is motivated by the fact that the geometric Cauchy problem for submanifolds of constant nullity r is in general not well-posed. More specifically, in order to ensure uniqueness of the solution, one needs to prescribe the tangent bundle along an r-dimensional submanifold of $\mathbb{R}^{m+n}$.

4.3.1 Paper E

At the time of submitting this thesis, Paper E is still work in preparation. I would like to complete it by giving a local description of the family of full developable submanifolds of $\mathbb{R}^{m+n}$. This will answer David Brander's interesting question on whether full developable submanifolds are plentiful or rare.

Although I am still lacking a satisfactory statement of the answer, I believe that they come in great abundance. Indeed, it is not difficult to see that any oriented full developable submanifold M^m can be described (in a neighborhood of a point) by $m+n-1$ smooth functions, n of which are non-vanishing.

To prove this claim, take a unit-speed curve $\gamma\colon I = [0,\alpha] \to M$ orthogonal to all the rulings. Since we are of course interested in describing M up to rigid motions of the ambient space, there is no loss of generality in assuming that $\dot\gamma(0) = e_1$, $T_{\gamma(0)}M = \operatorname{span}(e_1,\dots,e_m)$ and $N_{\gamma(0)}M = \operatorname{span}(e_{m+1},\dots,e_{m+n})$, where e_q denotes the q-th standard basis vector of $\mathbb{R}^{m+n}$. Then define an oriented orthonormal frame for

$T\mathbb{R}^{m+n}|_\gamma$ as follows:

1. Smoothly extend $(e_i)_{i=1}^m$ into a γ-adapted orthonormal frame $(E_i)_{i=1}^m$ for the tangent bundle of M along γ, such that $E_2, \dots, E_m$ are parallel in the normal connection of γ;

2. Smoothly extend $(e_{m+k})_{k=1}^n$ into an orthonormal frame $(N_k)_{k=1}^n$ for the normal bundle of M along γ, parallel in the normal connection of M.

   Now, since γ is orthogonal to the rulings, developability of M implies that $\overline{D}_t N_k \cdot E_2 = \cdots = \overline{D}_t N_k \cdot E_m = 0$ for every k. Moreover, for M is assumed to be full, $\overline{D}_t N_k \cdot E_1$ is non-vanishing for every k.

   Summing up, the chosen frame satisfies the linear ODE-problem

$$\begin{cases} \overline{D}_t E_1 = -\kappa_1 E_2 - \cdots - \kappa_{m-1} E_m - \tau_1 N_1 - \cdots - \tau_n N_n \\ \overline{D}_t E_2 = \kappa_1 E_1 \\ \vdots \\ \overline{D}_t E_m = \kappa_{m-1} E_1 \\ \overline{D}_t N_k = \tau_k E_1 \quad (k = 1, \dots, n) \\ (E_i(0), N_k(0))_{i,k=1}^{m,n} = (e_1, \dots, e_{m+n}) \end{cases}, \tag{4.1}$$

   where $(\kappa_j)_{j=1}^{m-1}$ and $(\tau_k)_{k=1}^n$ are tuples of smooth functions $I \to \mathbb{R}$. In particular, $\tau_k(t) \neq 0$ for every k and t.

   Conversely, for any arbitrary choice of $(\kappa_j)_{j=1}^{m-1}$ and $(\tau_k)_{k=1}^n$, problem (4.1) has unique global solution, thus defining a full developable submanifold up to a rigid motion of $\mathbb{R}^{m+n}$.

Bibliography

[1] Jürgen Berndt, Sergio Console, and Carlos Enrique Olmos, *Submanifolds and holonomy*, second ed., Monographs and Research Notes in Mathematics, CRC Press, Boca Raton, 2016.

[2] Shiing-shen Chern and Nicolaas H. Kuiper, *Some theorems on the isometric imbedding of compact Riemann manifolds in euclidean space*, Ann. of Math. (2) **56** (1952), 422–430.

[3] M. Dajczer, V. Rovenski, and R. Tojeiro, *Euclidean hypersurfaces with a totally geodesic foliation of codimension one*, Geom. Dedicata **176** (2015), 215–224.

[4] Marcos Dajczer and Detlef Gromoll, *Gauss parametrizations and rigidity aspects of submanifolds*, J. Differential Geom. **22** (1985), 1–12.

[5] Marcos Dajczer and Ruy Tojeiro, *Submanifolds with nonparallel first normal bundle*, Canad. Math. Bull. **37** (1994), 330–337.

[6] ———, *Submanifolds with nonparallel first normal bundle revisited*, Publ. Mat. **58** (2014), 179–191.

[7] Erik D. Demaine and Gregory N. Price, *Generalized D-forms have no spurious creases*, Discrete Comput. Geom. **43** (2010), 179–186.

[8] Manfredo P. do Carmo, *Differential geometry of curves and surfaces*, Prentice-Hall, Englewood Cliffs, 1976.

[9] Joseph Erbacher, *Reduction of the codimension of an isometric immersion*, J. Differential Geom. **5** (1971), 333–340.

[10] Luis A. Florit, *Parametrizations in submanifold theory*, Instituto de Matemática Pura e Aplicada (IMPA), Rio de Janeiro, 1999.

[11] Luis A. Florit and Guilherme M. de Freitas, *Classification of codimension two deformations of rank two Riemannian manifolds*, Comm. Anal. Geom. **25** (2017), 751–797.

[12] Mohammad Ghomi, *Open problems in geometry of curves and surfaces*, available at `www.math.gatech.edu/~ghomi/Papers/op.pdf`.

[13] Joseph Grifone and Jean-Marie Morvan, *External curvatures and internal torsion of a Riemannian submanifold*, J. Differential Geom. **16** (1981), 351–371.

[14] Philip Hartman and Louis Nirenberg, *On spherical image maps whose Jacobians do not change sign*, Amer. J. Math. **81** (1959), 901–920.

[15] Shyuichi Izumiya and Saki Otani, *Flat approximations of surfaces along curves*, Demonstr. Math. **48** (2015), 217–241.

[16] Shoshichi Kobayashi and Katsumi Nomizu, *Foundations of differential geometry*, vol. 1, Wiley-Interscience, New York, 1963.

[17] Snežana Lawrence, *Developable surfaces: Their history and application*, Nexus Network Journal **13** (2011), 701–714.

[18] John M. Lee, *Riemannian geometry: an introduction to curvature*, Graduate Texts in Mathematics, no. 176, Springer, New York, 1997.

[19] Katsumi Nomizu, *Kinematics and differential geometry of submanifolds*, Tohoku Math. J. (2) **30** (1978), 623–637.

[20] Helmut Pottmann and Johannes Wallner, *Computational line geometry*, Mathematics and Visualization, Springer-Verlag, Berlin, 2001.

[21] M. C. Romero-Fuster and V. D. Sedykh, *A lower estimate for the number of zero-torsion points of a space curve*, Beiträge Algebra Geom. **38** (1997), 183–192.

[22] Vladimir Y. Rovenskii, *Foliations on Riemannian manifolds and submanifolds*, Birkhäuser, Boston, 1998.

[23] Michael Spivak, *A comprehensive introduction to differential geometry*, third ed., vol. 4, Publish or Perish, Houston, 1999.

[24] Vitaly Ushakov, *Developable surfaces in Euclidean space*, J. Austral. Math. Soc. Ser. A **66** (1999), 388–402.

[25] Tony Wills, *D-Forms: 3D forms from two 2D sheets*, Bridges London: Mathematics, Music, Art, Architecture, Culture (Reza Sarhangi and John Sharp, eds.), Tarquin Publications, 2006, pp. 503–510.

Part II

Included papers

Sculpturing Surfaces with Cartan Ribbons

Matteo Raffaelli, Jakob Bohr, and Steen Markvorsen
DTU Compute and DTU Nanotech, Technical University of Denmark
matraf@dtu.dk, jabo@nanotech.dtu.dk, stema@dtu.dk

Abstract

Using the concepts of *Cartan development* and *rolling* from differential geometry we develop a method for sculpturing any surface with the use of Cartan ribbons.

Peeling and Pasting

When peeling an apple or a potato you may steer the peeler along any chosen curve. Examples are shown in Figure 1 together with the corresponding peelings. Here we initiate a geometric study of this operation and

Figure 1 : *Examples of steering curves and corresponding peelings of a potato and an apple*

its reverse: we construct the flat ribbons that can be pasted onto a given surface so that they approximate the surface to first, tangential, order along one or more curves on the surface. We can imagine that once the flat ribbons have been mathematically described, they can be cut out of paper, a thin metal plate, or any other flat, bendable material. The flat ribbons are constructed along the so-called *Cartan development* of the given surface curve – that is why we call these ribbons *Cartan ribbons*.

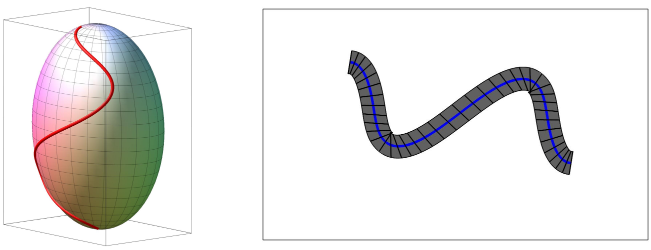

Figure 2 : *The result of a peeling along a given curve on an ellipsoid*

This pasting technique, the sculpting, with Cartan ribbons works along any given curve on the given sculpture surface as long as the curve just avoids the asymptotic directions of the surface. A good starting point for a precise discussion of the method and of this condition is the seminal paper by K. Nomizu [8]. Nomizu gives a kinematic interpretation of the Cartan development via the rolling of the given surface on a plane – see also the recent work by M. G. Molina, [7]. We can imagine that the designated curve on the surface is covered with wet paint. We then roll the surface on a plane without skidding and twisting, and in such a way that the points of contact between the surface and the plane are precisely along the given curve. Then the wet curve on the surface will trace out and paint a new curve on the plane. This new planar curve is precisely the Cartan development of the original surface curve. One intuitive way of understanding the procedure is then to divide it into three steps: (1) the given curve on the surface is rolled into the plane as described above; (2) the resulting Cartan development curve is extended to a strip, a Cartan ribbon, in the plane; and finally (3) the ribbon is rolled back along the development curve onto the surface. Using an ellipsoid to replace the potato, the three steps are illustrated in Figures 3, 4, and 5. The resulting flat peeling along the given curve is shown on the right in Figure 2, where the transversal rulings (like railroad sleepers) along the center curve are also shown. They represent the lines around which the Cartan ribbon must be bent in order to fit onto the surface along the given designated curve. The precise angle function θ of the rulings w.r.t. the curve tangent is discussed in the mathematics section below, see also [9] for a thorough discussion and generalizations.

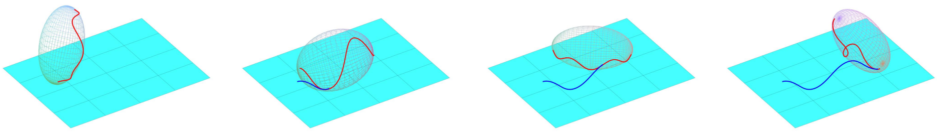

Figure 3 : *Step 1: Generating the Cartan development curve via rolling of the surface*

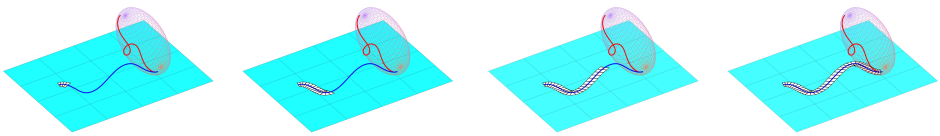

Figure 4 : *Step 2: Ribbonizing the Cartan development curve via the ruling angle function θ*

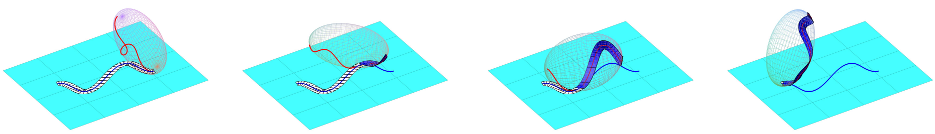

Figure 5 : *Step 3: Pasting the Cartan ribbon onto the surface via back-rolling*

This procedure clearly paves the way for a novel technique of approximating any surface or sculpture with one or more intrinsically flat Cartan ribbons. See [10, 6] for modern industrial applications of similar techniques. We can extend the ribbon until it intersects the extension of another ribbon (or itself); such surface approximations are illustrated in Figures 6 and 7.

From an artistic point of view the created surfaces have a surprising appearance. They contain wedges (ridges) as well as singly curved piecewise smooth and regular surfaces. They can be fabricated artistically in *coarse grained* versions creating unique surfaces with many obvious applications, e.g., lamp shades, chairs, roofs, facades, and the design of clothing and artworks, see [4, 5]. Or they can be *fine grained* in which case the underlying (typically *doubly curved*) surfaces become almost perfectly approximated by the *flat* Cartan ribbons as seen in the Figures 6 and 7. An interesting bonus is that the method also can be used to transform different surfaces into each other by using the same set of Cartan developments only with different angle functions θ for the rulings. Artistically, it is thus possible to make interesting intermediate versions of two approximated surfaces.

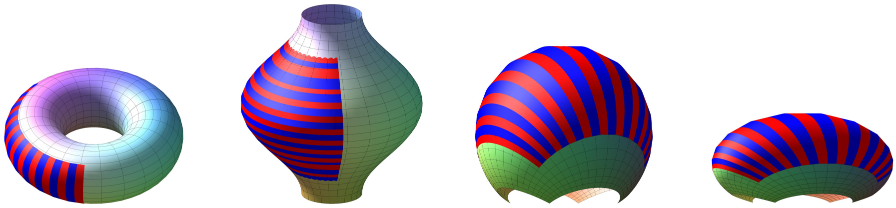

Figure 6 : *Surfaces with systems of approximating Cartan ribbons*

The Mathematics

The main geometric question in our setting is how precisely to cut the Cartan ribbons and how to find and describe the bending, i.e., the angle function θ, of a Cartan ribbon that will actually fold it onto the surface along the given surface curve. We answer these questions by surveying here the explicit recipes for both the cutting and for the bending. Details for this particular application as well as generalizations will appear in [9]. We also refer to [1, 2] for similar geometric results and other applications of ribbon geometry.

For the precise discussion of this recipe we need the notions of normal curvature κ_n, geodesic curvature κ_g, and geodesic torsion τ_g of the given designated curve on the surface. They are found most easily via the so-called Darboux frame adapted to the surface along γ. It consists of three orthogonal unit vector fields $\{e, h, N\}$ along γ. Here $N(t)$ denotes the unit normal vector to the surface at $\gamma(t)$ and $e(t) = \gamma'(t)/v(t)$ is the unit tangent vector of γ (where we have denoted the speed of γ by $v(t) = \|\gamma'(t)\|$), so that finally $h(t) = N(t) \times e(t)$ completes the orthonormal Darboux frame, see [3]. Then $v(t)\,\kappa_n(t) = e'(t) \cdot N(t)$, $v(t)\,\kappa_g(t) = e'(t) \cdot h(t)$, and $v(t)\,\tau_g(t) = h'(t) \cdot N(t)$. We can now state the result of Nomizu as follows:

Theorem [8, Nomizu, Theorem 2 p. 630] *Let γ denote a smooth regular curve on a given surface M. Suppose that the normal curvature $\kappa_n(t)$ is never zero along γ. Then there exists a unique rolling Rol_t of M on the (x, y)-plane (without skidding and without spinning) such that $\eta(t) = \mathrm{Rol}_t(\gamma(t))$ (with $\eta(0) = (0, 0)$ and $\eta'(0) = (0, \|\gamma'(0)\|)$) is the locus of points of contact during the rolling. The curve η is the Cartan development of the given curve γ.*

The condition that *the curve just avoids the asymptotic directions of the surface* alluded to above is here encoded into the assumption that *the normal curvature $\kappa_n(t)$ is never zero along γ*. Such a condition is needed: a non-flat ruled surface, for example a hyperboloid of one sheet, cannot roll in the direction of its rulings, so the curve γ on the surface should never have a tangent parallel to such a ruling. Correspondingly, the Cartan ribbon construction also needs special care when $\kappa_n(t)$ is 0:

Proposition [9] *The unique flat (but bent) Cartan ribbon along γ which has the same normal field N as the surface M is well defined for $\kappa_n(t) \neq 0$. It is the ruled, developable, surface determined by the ingredients $e(t)$, $h(t)$, $\kappa_n(t)$, and $\tau_g(t)$ as follows: $r(t, u) = \gamma(t) + u \cdot \left(\frac{\tau_g(t)}{\kappa_n(t)} \cdot e(t) + h(t) \right)$, where $u \in [-w, w]$.*

In this representation the ribbon has constant width $2w$; it can easily be modified to variable widths $w_+(t)$ to the left hand side (the $h(t)$ side) of γ and $w_-(t)$ to the right hand side (the $-h(t)$ side) of γ just by substituting the u interval by $u \in [-w_-(t), w_+(t)]$. As is evident from the examples in Figures 6 and 7 we need to apply this type of restriction so that the approximating ribbons line up along their edges.

The tangent vectors $\gamma'(t)$ and $\eta'(t)$ have the same coordinates with respect to corresponding parallel frames along the curves on M and in the plane, respectively. The angle function θ of the rulings is therefore determined by the angle between the vector $(\tau_g(t)/\kappa_n(t)) \cdot e(t) + h(t)$ and the tangent vector $e(t)$, i.e., $\cot(\theta(t)) = \tau_g(t)/\kappa_n(t)$. The bending angle variation is illustrated on the right in Figure 2. The angle is well-defined and 0 precisely when $\kappa_n(t) = 0$ which must be avoided, because in such a case the ruling is directed in the tangent direction of the curve and the given parametrization of the ribbon is thence not regular. In this parametrization, then, the condition for regularity is the same as Nomizu's condition for the existence of a rolling, namely $\kappa_n(t) \neq 0$. Moreover, in order to obtain regularity of the ribbon with a finite width, the width to the *curvature side* of the center curve must also be less than $1/\kappa_g(t)$.

It follows in particular from Nomizu's theorem that the tangent vectors $\gamma'(t)$ and $\eta'(t)$ have the same length for all t and that the two curves have the same geodesic curvature function κ_g on M and in the plane respectively. This latter property means that if we are given one of the curves, then the other curve can be explicitly constructed by solving the ordinary differential equation system that produces the curve from its geodesic curvature function, see [3]. In this way the curve and its development are dual constructions – the given curve may be considered and re-constructed on the surface in this way as the *anti-development* of the Cartan development curve. We note in particular, that in the case of M being a sphere of any radius ρ, then $\kappa_n(t) = 1/\rho$ and $\tau_g(t) = 0$ along every curve in M, so we get $\theta(t) = \pi/2$ for all t. This makes it particularly simple and easy to construct the ribbons along curves on spheres, see Figure 7.

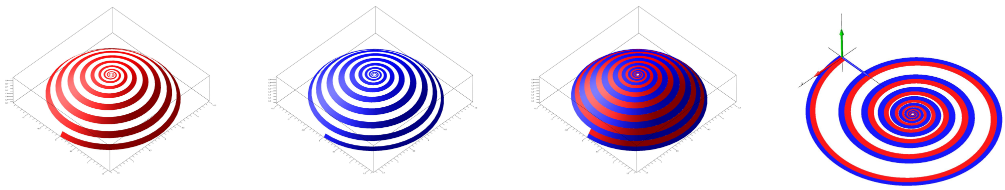

Figure 7 : *A Cartan ribbon (with controlled widths $w_{\pm}$) covers the spherical cap along a given spiral*

Acknowledgement. The authors would like to thank the referees for their careful reading of the manuscript.

References

[1] J. Bohr and S. Markvorsen: *Ribbon Crystals*. PLoS ONE 8(10): e74932 (2013).

[2] J. Bohr and S. Markvorsen: *Autorotation*. Physica Scripta **91** 023005 (2016).

[3] M. P. do Carmo: *Differential geometry of curves and surfaces*. Pearson, 1976.

[4] K. Delp and B. Thurston: *Playing with Surfaces: Spheres, Monkey Pants, and Zippergons*, Bridges 2011: Mathematics, Music, Art, Architecture, Culture, the 14th Annual Bridges Conference in the University of Coimbra, Portugal, 2011. Eds.: Reza Sarhangi and Carlo H. Séquin. ISBN: 978-0-9846042-6-5. Tessellations Publishing, 2011.

[5] C. Goodman-Strauss: *Gyring Gyroid*, http://schoengeometry.com/e-tpms-media/gyringGyroid.pdf

[6] P. Mehrtens and M. Schneider: *Bahn frei für die Architektur – Approximation von Freiform-Flächen durch abwickelbare Streifen*, Stahlbau **81** (2012) Heft 12, 931–934.

[7] M. G. Molina: *Geometric conditions for the existence of a rolling without twisting or slipping*, Communications on pure and applied Analysis, Volume **13** (2014) 435–452.

[8] K. Nomizu: *Kinematics and differential geometry of submanifolds*, Tohoku Math. Journ. 30 (1978), 623–637.

[9] M. Raffaelli, J. Bohr, and S. Markvorsen: *Cartan Ribbons*. In preparation.

[10] A. Søndergaard et al.: *Robotic hot-blade cutting: an industrial approach to cost-effective production of double curved concrete structures.* Pp. 150–164 in Proceedings of RobArch 2016, Robotic Fabrication in Architecture, Art and Design 2016. Editors: D. Reinhardt, R. Saunders, and J. Burry. Springer 2016.

This is an Author Accepted Manuscript of an article published in *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, available online at: https://doi.org/10.1098/rspa.2017.0389.

CARTAN RIBBONIZATION AND A TOPOLOGICAL INSPECTION

JAKOB BOHR, STEEN MARKVORSEN, AND MATTEO RAFFAELLI

ABSTRACT. We develop the concept of Cartan ribbons together with a rolling-based method to ribbonize and approxi-mate any given surface in space by intrinsically flat ribbons. The rolling requires that the geodesic curvature along the contact curve on the surface agrees with the geodesic curvature of the correspon-ding Cartan development curve. Essentially, this follows from the orientational alignment of the two co-moving Darboux frames during rolling. Using closed contact center curves we obtain closed approximating Cartan ribbons that contribute zero to the total curvature integral of the ribbonization. This paves the way for a particularly simple topological inspection – it is reduced to the question of how the ribbons organise their edges relative to each other. The Gauss-Bonnet theorem leads to this topological inspection of the vertices. Finally, we display two examples of ribbonizations of surfaces, namely of a torus using two ribbons, and of an ellipsoid using closed curvature lines as center curves for the ribbons.

1. Introduction

The approximation of surfaces by patch-works of planar parts has a long use in fundamental and applied mathematics. Foremost comes to mind the multifaceted applications of triangulations [1]. In the present work we develop a scheme for approximating a surface by the use of multiple developable surfaces. Some of the beauty of this approach is the relatively few numbers of developable stretches – ribbons – needed to approximate a given surface. Not to mention that the study of shapes and structures of developable surfaces is itself a classical subject that has intrigued mathematicians for centuries and has found numerous artistic applications in architecture and design, see [2].

In the seventies K. Nomizu pointed out that the concept of (extrinsic) rolling can be understood as a kinematic interpretation of the (intrinsic) Levi-Civita connection and of the Cartan development of curves, see [3] and [4]. One derives simple expressions for the components of the corresponding relative angular velocity vector of the rolling, i.e. the geodesic torsion, the normal curvature, and the geodesic curvature of the given curve and its development, see [5, 6, 7, 9, 8]. For example, in conjunction with a plane, the rolling must propagate along a planar curve which has the same geodesic curvature as the given curve, see examples in [10].

In recent years rolling has received a renewed wave of interest – in part because of its importance for robotic manipulation of objects [11]. For example, there has been an interest in understanding rolling from symmetry arguments [12] as well as purely geometrical considerations [13, 14, 15]. Also, the shapes known as D-forms are examples of surface structures that are formed by assembling several developable surfaces [16, 17, 18].

The paper is organized as follows: In sections 2 and 3 we apply the notion of rolling as an alternative entrance to the construction of developable surface approximations. We show how the method of rolling a surface along the planar Cartan development of a given curve on the surface produces a planar ribbon which – after isometric bending along the lines of the instantaneous rotation axes – will reproduce the surface approximation along the said curve. In other words, the rolling induces a local isometry between the flat approximation along the curve and the plane. Further in section 3 we discuss a specific measure of the local goodness of a given ribbon approximation. In section 4 we then initiate the corresponding study of such approximations by establishing a precise calculation of the Euler characteristic of the surfaces via an inspection of the family of approximating ribbons. Finally, in sections 5 and 6, we illustrate the approximation method by two concrete examples which show the ensuing Cartan ribbon approximations of a torus (along two trigonometric center curves) and of an ellipsoid (along six lines of curvature), respectively.

2. The Initial Setting

We consider two surfaces S and $\tilde{S}$ in $\mathbb{R}^3$. Let γ be a smooth, regular curve on S, $\gamma : J = [0, \alpha] \to S$, such that $\gamma(0) = (0, 0, 0)$. We equip γ with its *Darboux frame field* $\mathcal{F} = \{e, h, N\}$, defined as follows: for each $t \in J$ we let $N(t)$ denote a unit normal vector to S at $\gamma(t)$, we let $e(t) = \gamma'(t)/\|\gamma'(t)\|$ the unit tangent vector of γ and $h(t) = N(t) \times e(t)$. The frame $\mathcal{F}$ then satisfies the following equations – see for example [19, Corollary 17.24]:

$$\begin{bmatrix} e'(t) \\ h'(t) \\ N'(t) \end{bmatrix} = \|\gamma'(t)\| \cdot \begin{bmatrix} 0 & \kappa_g(t) & \kappa_n(t) \\ -\kappa_g(t) & 0 & \tau_g(t) \\ -\kappa_n(t) & -\tau_g(t) & 0 \end{bmatrix} \begin{bmatrix} e(t) \\ h(t) \\ N(t) \end{bmatrix} , \tag{1}$$

where $\tau_g(t)$, $\kappa_n(t)$, and $\kappa_g(t)$ are the geodesic torsion, the normal curvature, and the geodesic curvature, respectively, of γ at $\gamma(t)$. Since we are so far only considering local geometric entities, the surfaces S and $\tilde{S}$ need not be orientable, i.e. the frame $\mathcal{F}$ and its properties – such as the signs appearing in (1) – depend on the local choice of normal vector field N. In the final sections we will note a few consequences concerning the rolling and the corresponding ribbonization of non-orientable surfaces.

Moving S on $\tilde{S}$. Given a curve γ on S as above, we now consider smooth and regular curves $\tilde{\gamma}$ on the other surface $\tilde{S}$ such that the following initial compatibility and contact conditions are satisfied:

$$\begin{aligned} \tilde{\gamma}(0) &= \gamma(0) = (0, 0, 0) \\ \tilde{\gamma}'(0) &= \gamma'(0) \\ \|\tilde{\gamma}'\| &= \|\gamma'\| \quad , \end{aligned} \tag{2}$$

so that $\tilde{\gamma}$ has the same initial point and direction as γ and so that $\tilde{\gamma}$ has the same speed as γ for all $t \in J$. A framed motion of (S, γ) on $\tilde{S}$ is then defined as follows:

Definition 2.1. *Let $E^+(3)$ be the group of direct isometries of* $\mathbb{R}^3$. *A (1*-parameter*)* framed motion g_t of (S, γ) on $\tilde{S}$ along $\tilde{\gamma}$ *is a differentiable map* $J \to E^+(3)$ *such that for each t the map g_t is the isometry that maps*

$$\begin{aligned} \gamma(t) \ &\text{to}\ \tilde{\gamma}(t), \\ \gamma(t) + e(t) \ &\text{to}\ \tilde{\gamma}(t) + \tilde{e}(t), \\ \gamma(t) + N(t) \ &\text{to}\ \tilde{\gamma}(t) + \tilde{N}(t) \quad , \end{aligned} \tag{3}$$

where $\tilde{e}$ and $\tilde{N}$ are two of the members of the Darboux frame $\tilde{\mathcal{F}} = \{\tilde{e}\,,\ \tilde{h} = \tilde{N} \times \tilde{e}\,,\ \tilde{N}\}$ along $\tilde{\gamma}$ on $\tilde{S}$ defined in the same way as the frame $\mathcal{F}$ along γ on S. The point $\tilde{\gamma}(t)$ is called the contact point *at instant t, and $\tilde{\gamma}(J)$ is called the* contact curve *of the framed motion g_t of (S, γ) on $\tilde{S}$.*

Since g_t is in particular an instantaneous isometry it is represented by $x \mapsto R_t x + c_t$, where $R_t \in SO(3)$ is a rotation matrix and c_t a translation vector. The instantaneous framed motion is then given by the vector field $V_t : x \mapsto \Omega_t(x - c_t) + c_t'$, with $\Omega_t = R_t' R_t^{\mathsf{T}}$, see [3]. As g_t is a *framed motion* we have:

Proposition 2.2. *Let D_t be the matrix having $e(t)$, $h(t)$ and $N(t)$ as coordinate column vectors (with respect to a fixed coordinate system in $\mathbb{R}^3$) and similarly, let $\tilde{D}_t$ be the matrix having $\tilde{e}(t)$, $\tilde{h}(t)$ and $\tilde{N}(t)$ as coordinate column vectors (with respect to the same fixed coordinate system in $\mathbb{R}^3$). Then*

$$\begin{aligned} R_t &= \tilde{D}_t D_t^{\mathsf{T}} \\ c_t &= \tilde{\gamma}(t) - R_t \gamma(t) \quad , \end{aligned} \tag{4}$$

so that

$$g_t(x) = \tilde{D}_t D_t^{\mathsf{T}}(x - \gamma(t)) + \tilde{\gamma}(t) \quad . \tag{5}$$

Proof. The rotation $\tilde{D}_t D_t^{\mathsf{T}}$ maps the vector $e(t)$ to $\tilde{e}(t)$, and $N(t)$ to $\tilde{N}(t)$. The representation $g_t(x) = R_t x + c_t$ is therefore given by (5). □

Rolling S on $\tilde{S}$. A framed motion g_t of (S, γ) on $\tilde{S}$ along $\tilde{\gamma}$ is said to be *rotational* if, for all $t \in J$, Ω_t is different from the zero matrix. At each time instant we can then find a unique vector $\omega_t \neq 0$, the *angular velocity vector*, such that $\omega_t \times x = \Omega_t x$ for all $x \in \mathbb{R}^3$.

Based on the orientation of the angular velocity vector relative to the common tangent plane of $g_t(S)$ and $\tilde{S}$, we introduce the following terminology for the instantaneous motion – which extends directly to the entire motion.

Definition 2.3. *The instantaneous rotational framed motion g_t is a* pure spinning *if the angular velocity vector ω_t is orthogonal to the tangent plane $T_{\tilde{\gamma}(t)}\tilde{S}$, and a* pure twisting *if ω_t is proportional to the tangent vector $\tilde{e}(t)$. Finally, the motion g_t will be called a* standard rolling *if ω_t does not contain a spinning component and is not a pure twisting, i.e. a standard rolling of S on $\tilde{S}$ is characterized by the condition that there exist smooth functions a and b such that ω_t decomposes as follows for all t:*

$$\omega_t = a(t) \cdot \tilde{e}(t) + b(t) \cdot \tilde{h}(t) + 0 \cdot \tilde{N}(t) \quad , \quad b(t) \neq 0 \quad . \tag{6}$$

It turns out that a standard rolling of a given surface S on a *plane* gives a kinematic approach towards the construction of approximating developable ribbons that is presented below in section 3. To begin with, we observe the following result for the more general situation of rolling S on a general surface $\tilde{S}$:

Proposition 2.4. *With the setting introduced above, a framed motion g_t of (S,γ) on $\tilde{S}$ along $\tilde{\gamma}$ is a standard rolling if and only if the following conditions are satisfied for all $t \in J$:*

$$\begin{aligned} \kappa_g(t) &= \tilde{\kappa}_g(t) \\ \kappa_n(t) &\neq \tilde{\kappa}_n(t) \quad , \end{aligned} \tag{7}$$

where $\tilde{\kappa}_g$ and $\tilde{\kappa}_n$ denote the geodesic curvature and the normal curvature of $\tilde{\gamma}$, respectively.

Proof. As in proposition 2.2, $R_t = \tilde{D}_t D_t^{\mathsf{T}}$ and $c_t = \tilde{\gamma}(t) - R_t\gamma(t)$. Then, $g_t(x) = R_t x + c_t$, and so we can find the instantaneous motion V_t by computing $\Omega_t(x - c_t) + c_t'$. Since $c_t' = \tilde{\gamma}'(t) - R_t'\gamma(t) - R_t\gamma'(t) = -R_t'\gamma(t)$ for R_t maps $\gamma'(t)$ to $\tilde{\gamma}'(t)$, we obtain

$$\begin{aligned} V_t(x) &= \Omega_t(x - \tilde{\gamma}(t) + R_t\gamma(t)) - R_t'\gamma(t) \\ &= \Omega_t x - \Omega_t\tilde{\gamma}(t) + R_t'\gamma(t) - R_t'\gamma(t) \\ &= \Omega_t(x - \tilde{\gamma}(t)), \end{aligned} \tag{8}$$

where $\Omega_t = R_t' R_t^{\mathsf{T}} = \tilde{D}_t'\tilde{D}_t + \tilde{D}_t D_t'^{\mathsf{T}} D_t \tilde{D}_t^{\mathsf{T}}$. If now we let

$$\varLambda_t = \|\gamma'(t)\| \begin{bmatrix} 0 & \kappa_g(t) & \kappa_n(t) \\ -\kappa_g(t) & 0 & \tau_g(t) \\ -\kappa_n(t) & -\tau_g(t) & 0 \end{bmatrix}, \tag{9}$$

we have – from (1) – that $\tilde{D}_t' = \tilde{D}_t\tilde{\varLambda}_t^{\mathsf{T}} = -\tilde{D}_t\tilde{\varLambda}_t$ ($\tilde{\varLambda}_t$ is skew symmetric) as well as $D_t'^{\mathsf{T}} D_t = \varLambda_t$. Hence, if $\varXi_t = \varLambda_t - \tilde{\varLambda}_t$, that is

$$\varXi_t = \begin{bmatrix} 0 & \varXi_t^{1,2} & \varXi_t^{1,3} \\ -\varXi_t^{1,2} & 0 & \varXi_t^{2,3} \\ -\varXi_t^{1,3} & -\varXi_t^{2,3} & 0 \end{bmatrix}, \tag{10}$$

where

$$\begin{aligned} \varXi_t^{1,2} &= \|\gamma'(t)\| \cdot (\kappa_g(t) - \tilde{\kappa}_g(t)) \\ \varXi_t^{1,3} &= \|\gamma'(t)\| \cdot (\kappa_n(t) - \tilde{\kappa}_n(t)) \\ \varXi_t^{2,3} &= \|\gamma'(t)\| \cdot (\tau_g(t) - \tilde{\tau}_g(t)), \end{aligned} \tag{11}$$

the expression for Ω_t reduces to

$$\Omega_t = \tilde{D}_t \varXi_t \tilde{D}_t^{\mathsf{T}}, \tag{12}$$

and the resulting angular velocity vector of the rolling is thence – with respect to the Darboux frame $\tilde{\mathcal{F}}(t) = \{\tilde{e}(t), \tilde{h}(t), \tilde{N}(t)\}$ along $\tilde{\gamma}$ in $\tilde{S}$:

$$\begin{aligned} \omega_t &= \left(-\varXi_t^{2,3}\,,\ \varXi_t^{1,3}\,,\ -\varXi_t^{1,2}\right)_{\tilde{\mathcal{F}}(t)} \\ &= \|\gamma'(t)\| \cdot (-\tau_g(t) + \tilde{\tau}_g(t)\,,\ \kappa_n(t) - \tilde{\kappa}_n(t)\,,\ -\kappa_g(t) + \tilde{\kappa}_g(t))_{\tilde{\mathcal{F}}(t)}\ . \end{aligned} \tag{13}$$

By comparing (13) with (6) we see that the conditions (7) are necessary and sufficient for g_t to be a standard rolling. □

In passing we note – for later use – that (13) and proposition 2.4 immediately give the coordinates of the pulled-back angular rotation vector $\hat{\omega}_t = R_t^{\mathsf{T}} \omega_t$ with respect to the frame $\mathcal{F}(t)$ for a standard rolling:

$$\hat{\omega}_t = \|\gamma'(t)\| \cdot (-\tau_g(t) + \tilde{\tau}_g(t)\,,\ \kappa_n(t) - \tilde{\kappa}_n(t)\,,\ 0)_{\mathcal{F}(t)}\ . \tag{14}$$

The important special case in which $\tilde{S}$ is a plane is covered by the following corollary:

Corollary 2.5. *If $\tilde{S}$ is a plane, then the motion g_t is a standard rolling if and only if*

$$\begin{aligned} \kappa_g(t) &= \tilde{\kappa}_g(t) \\ \kappa_n(t) &\neq 0 \quad . \end{aligned} \tag{15}$$

The instantaneous angular rotation vector ω_t and its pull-back $\hat{\omega}_t$ are correspondingly – in $\tilde{\mathcal{F}}(t)$ and $\mathcal{F}(t)$ respectively:

$$\begin{aligned} \omega_t &= \|\gamma'(t)\| \cdot (-\tau_g(t), \kappa_n(t), 0)_{\tilde{\mathcal{F}}(t)} \\ &= \|\gamma'(t)\| \cdot (-\tau_g(t), \kappa_n(t), 0)_{\mathcal{F}(t)} \quad , \end{aligned} \tag{16}$$

where now $\tilde{\mathcal{F}}(t) = \{\tilde{e}(t), \tilde{e}_3 \times \tilde{e}(t), \tilde{e}_3\}$ is the co-moving frame in the plane with constant normal vector field $\tilde{e}_3$ along $\tilde{\gamma}$.

3. Developable Cartan surface ribbons

In this section we show that the rolling discussed above serves as a tool for obtaining a flat developable approximation of the surface S along γ. This is alternative to constructing developable approximations via envelopes of tangent planes along γ, see [20, pp. 195-197]. In the recent work [8] osculating developable surfaces and their singularities have been studied, see also [21]. It will follow from the condition (15) that the approximating surface is free of singularities in a neighbourhood of γ, see theorem 3.1 below.

We first consider the notion of ruled surfaces, since developable surfaces constitute a special subcategory of those:

Let w_- and w_+ denote two positive functions on the given t-interval J, let $I = [-w_-(t), w_+(t)]$, and let V denote the corresponding parameter domain in $\mathbb{R}^2$. A *parametrized ruled surface (with boundary)* $r\colon V \to \mathbb{R}^3$ based on the center curve γ is determined by a non-vanishing vector field β along γ:

$$r(t,u) = \gamma(t) + u \cdot \beta(t) \quad , \quad t \in J, \ \ u \in I. \tag{17}$$

We will assume that β is a unit vector field along γ and that the surface r is regular, i.e. its partial derivatives are linearly independent for all u in the interval $[-w_-(t), w_+(t)]$, $t \in J$. Regularity implies in particular that

$$\beta(t) \neq \pm e(t) \text{ for all } t \in J. \tag{18}$$

Moreover, the surface $r(V)$ is flat (with Gaussian curvature zero at all points, i.e. developable), precisely when the following condition is satisfied – see [20, p.

194]:

$$\beta' \cdot (\beta \times e) = 0. \tag{19}$$

If $r(V)$ is eventually to be constructed so that it becomes a flat approximation of S along γ, we need to find a regular parametrization r such that $r(V)$ is developable and has the same normal field N as S along γ. It means that we need to determine the vector function β so that it fulfills (18), (19), and

$$\beta \cdot N = 0. \tag{20}$$

The desired vector function β is precisely (modulo length and sign) the previously encountered pulled-back angular velocity vector $\hat{\omega}_t$ along γ associated with the rolling of S along $\tilde{\gamma}$ on a plane, see [10]:

Theorem 3.1. *Let γ denote a smooth curve on a surface S and let $\mathcal{F} = \{e, h, N\}$ be the corresponding Darboux frame field along γ. Suppose that the normal curvature function κ_n for γ on S never vanishes. Then there exists a unique developable surface which contains γ and which has everywhere the same tangent plane as S along γ. It is parametrized as follows:*

$$r(t, u) = \gamma(t) + u \cdot \frac{\hat{\omega}_t}{\|\hat{\omega}_t\|} \quad , \quad u \in [w_-(t), w_+(t)] \quad , \quad t \in J. \tag{21}$$

where $\hat{\omega}_t$ denotes the pulled-back angular velocity vector:

$$\begin{aligned} \hat{\omega}_t &= \kappa_n(t) \cdot h(t) - \tau_g(t) \cdot e(t) \quad , \\ \|\hat{\omega}_t\| &= \sqrt{\kappa_n^2(t) + \tau_g^2(t)} \quad . \end{aligned} \tag{22}$$

Proof. Write β in terms of its coordinate functions $\beta \cdot e$ and $\beta \cdot h$, substitute into equation (19) and apply equation (1) to express the derivatives of e and h. Then

$$\frac{\beta \cdot e}{\beta \cdot h} = -\frac{\tau_g}{\kappa_n} \quad , \tag{23}$$

and the result follows upon normalization of the solution β. The ruling directions of the developable surface are thus given by the instantaneous angular velocity vector of the rolling. □

Definition 3.2. *The developable surface, which is parametrized by* (21) *– and which is therefore approximating the surface S – will be called the* Cartan surface ribbon *along γ on S.*

As is already in the name, the Cartan surface ribbon can be *developed* isometrically into a planar ribbon:

Definition 3.3. *The* associated Cartan planar ribbon *for γ on S – which is defined along $\tilde{\gamma}$ in the plane – is now determined via* (24) *in the proposition below, which also establishes the isometry between the two Cartan ribbons.*

Proposition 3.4. *An isometry from the Cartan surface ribbon onto the associated Cartan planar ribbon is realized along the development curve $\tilde{\gamma}$ in the following way, which is in precise accordance with the previously found rolling of S*

along γ on the plane with contact curve $\tilde{\gamma}$. We simply map the point $r(t,u)$ to the point

$$\begin{aligned} \tilde{r}(t,u) &= \tilde{\gamma}(t) + u \cdot \frac{\omega_t}{\|\omega_t\|} \\ &= \tilde{\gamma}(t) + u \cdot \frac{\omega_t}{\sqrt{\kappa_n^2(t) + \tau_g^2(t)}} \quad . \end{aligned} \tag{24}$$

Proof. We let $\beta(t) = \omega_t/\|\omega_t\|$ and $\hat{\beta}(t) = \hat{\omega}_t/\|\hat{\omega}_t\|$. Since $\kappa_g(t) = \tilde{\kappa}_g(t)$ all the scalar products between two vectors chosen from $\{\gamma'(t), \hat{\beta}(t), \hat{\beta}'(t)\}$ are the same as the scalar products between the corresponding two vectors chosen from $\{\tilde{\gamma}'(t), \beta(t), \beta'(t)\}$. It follows that the two first fundamental forms for $r(t,u)$ and $\tilde{r}(t,u)$ respectively, have identical coordinate functions. The two ribbons r and $\tilde{r}$ are therefore isometric. □

Remark 3.5. *In all of the above constructions we have assumed that the center curves in question have nowhere vanishing normal curvature. For a number of cases the normal curvature does vanish, such as on planar faces of polyhedra and through lines of inflections on generalized cylindrical faces. The method of approximation by ribbons can be extended to these surfaces by cut and paste along the singular rulings under the condition that the geodesic torsion also vanishes together with the normal curvature. For example, for surfaces containing planar domains, the ribbonization can be continued over any edge of the planar domain if the ruling of the ribbon agrees with the given edge. For polyhedral surfaces this is always possible. A ribbon with planar patches will also be denoted a Cartan ribbon, see the later section on Euler's polyhedral formula.*

Curvature and parallel transport. In view of our observations concerning the rolling of S on the plane, it now makes sense to say that the Cartan surface ribbon can be *rolled* isometrically onto the associated Cartan planar ribbon. This is induced in the way just described by the rolling of S on the plane, which itself is represented by the pulled-back angular velocity vector field $\hat{\omega}$ along γ in S and by ω along $\tilde{\gamma}$ in the plane. Accordingly, once the center curve $\tilde{\gamma}$ in the plane has been constructed using $\tilde{\kappa}_g(t) = \kappa_g(t)$, then the approximating Cartan *surface* ribbon can be obtained via the inverse rolling of the Cartan *planar* ribbon backwards into contact with the surface S along γ. An early hint of this connection is presented in [22, pp. 227-228].

The key object for the actual construction of the approximating Cartan surface ribbon along a given curve γ on S is thence the planar curve $\tilde{\gamma}$, which may itself be constructed either by rolling, or – simpler – by integrating the curvature function κ_g of γ, but in the plane, in the well known way, see [20]:

Proposition 3.6. *Suppose $\tilde{\gamma}$ has (signed) curvature κ_g and speed $\|\tilde{\gamma}'\| = v$. Then, modulo rotation and translation in the plane, we have:*

$$\tilde{\gamma}(t) = \int_0^t v(\hat{t}) \cdot (\cos(\varphi(\hat{t}))\,,\ \sin(\varphi(\hat{t})))\, d\hat{t} \tag{25}$$

where

$$\varphi(\hat{t}) = \int_0^{\hat{t}} v(\hat{u}) \cdot \kappa_g(\hat{u})\, d\hat{u} \quad . \tag{26}$$

The curve $\tilde{\gamma}$ appears as a special – and simple – example of a *Cartan development* as already alluded to via the reference to Nomizu's initial work, see [3]. This is why the ensuing developable ribbons are called *Cartan surface ribbons*. To be a bit more specific concerning our simple 2-dimensional setting, we recall in particular the important geodesic curvature equivalence used above:

We let the tangent space $T_{\gamma(0)}S$ at $\gamma(0)$ represent the plane $\tilde{S}$ into which we want to construct the Cartan development curve corresponding to the given curve γ in S. For each t we consider the parallel transport of the tangent vector $\gamma'(t)$ along γ from the point $\gamma(t)$ to the point $\gamma(0)$, see [4, p. 131]:

$$X(t) = \Pi_{\gamma}^{\gamma(t)\,,\,\gamma(0)}\left(\gamma'(t)\right) \quad . \tag{27}$$

The Cartan development $\tilde{\gamma}$ of γ in $T_{\gamma(0)}S$ is then:

$$\tilde{\gamma}(t) = \int_0^t X(u)\;du \quad . \tag{28}$$

From this construction it follows in particular that

Proposition 3.7. *Any tangent vector $\tilde{\gamma}'(t_1) = X(t_1)$ is itself parallelly transported (in the usual Euclidean sense) along $\tilde{\gamma}$ in the tangent space $T_{\gamma(0)}S$ (which may be canonically identified with $T_{\tilde{\gamma}(0)}\tilde{S}$) from $(0,0)$ to $\tilde{\gamma}(t_1)$ and the (geodesic) curvature function of the planar curve $\tilde{\gamma}$ is equal to the geodesic curvature function of the original curve γ in S:*

$$\tilde{\kappa}_g(t) = \kappa_g(t) \quad \textit{for all } t. \tag{29}$$

Proof. Suppose Y is any parallel vector field along the curve γ on the surface S, then the angle $\theta(t) = \angle(Y(t), \gamma'(t))$ gives the geodesic curvature of γ via $\theta'(t) = \kappa_g(t)$. Since the same holds true by construction along the development curve $\tilde{\gamma}$ in the tangent plane, we get $\tilde{\theta}(t) = \theta(t)$ so that $\tilde{\kappa}_g = \kappa_g$. □

A measure of local goodness of Cartan ribbon approximations. A measure of the goodness of a single ribbon approximation along a given center curve γ can be obtained from the following construction. Close to γ the surface S can be parametrized as a graph surface 'over' the Cartan ribbon in the direction of *the normal field N of the ribbon* as follows:

(30)

$$S_\varepsilon \;:\; \sigma(t,u) = \gamma(t) + u\cdot\left(\frac{\kappa_n(t)h(t) - \tau_g(t)e(t)}{\sqrt{\kappa_n^2(t) + \tau_g^2(t)}}\right) + f(t,u)\cdot N(t)\,,\; t\in J\,,\; u\in[-\varepsilon,\varepsilon]\,,$$

where f denotes the corresponding 'height' function and ε is everywhere smaller than each of the width functions w_- and w_+ for all $t \in J$ along γ. (Both width functions have positive minima since they are positive and J is closed.) The function f clearly has $f(t,0) = f'(t,0) = 0$ for all $t \in J$, so that

$$f(t,u) = \frac{1}{2}f''(t,0)\cdot u^2 + O(u^3) \text{ for each } t\in J \text{ and for all } u\in[-\varepsilon,\varepsilon] \quad . \tag{31}$$

The domain in space that is enclosed 'between' the surface S_ε and the Cartan Ribbon is thence parametrized as follows:
(32)

$$\mathcal{D}_\varepsilon \,:\, R(t,u,w) = \gamma(t) + u \cdot \left(\frac{\kappa_n(t)h(t) - \tau_g(t)e(t)}{\sqrt{\kappa_n^2(t) + \tau_g^2(t)}} \right) + w \cdot f(t,u) \cdot N(t) \quad ,$$
$$\text{where } t \in J \;,\; u \in [-\varepsilon, \varepsilon] \;,\; w \in [0,1] \quad .$$

Definition 3.8. *We consider the volume of the domain $\mathcal{D}_\varepsilon$ as a natural local measure of goodness $\mathcal{M}(\gamma, \varepsilon)$ of our approximation of the surface S, i.e. of the approximation by the single Cartan ribbon to the tubular neighborhood $\mathcal{S}_\varepsilon$ of width 2ε along the center curve γ:*

$$\mathcal{M}(\gamma, \varepsilon) = \mathrm{Vol}(\mathcal{D}_\varepsilon) = \int_J \int_{-\varepsilon}^{\varepsilon} \int_0^1 |(R'_t \times R'_u) \cdot R'_w| \; dt\, du\, dw \quad . \tag{33}$$

We then have the following evaluation of $\mathcal{D}_\varepsilon$.

Theorem 3.9. *The goodness $\mathcal{M}(\gamma, \varepsilon)$ of the single ribbon approximation along a unit speed center curve γ can be expressed in terms of the curvature functions $H(t)$, $K(t)$, $\kappa_n(t)$ and $\tau_g(t)$ along γ as follows:*

$$\mathcal{M}(\gamma, \varepsilon) = \frac{1}{3}\varepsilon^3 \cdot \int_J F(H(t), K(t), \kappa_n(t), \tau_g(t))\, dt + O(\varepsilon^4) \quad , \tag{34}$$

where
(35)

$$F(H, K, \kappa_n, \tau_g) = \frac{\kappa_n^2}{\left(\kappa_n^2 + \tau_g^2\right)^{3/2}} \cdot \left| \left(\tau_g^2 - \kappa_n^2 + 2H\kappa_n - 2\tau_g \sqrt{2H\kappa_n - K - \kappa_n^2} \right) \right| \quad .$$

Proof. Using the parametrization of $\mathcal{D}_\varepsilon$ and the derivatives of the Darboux frame in (1) we find that the volume element $|(R'_t \times R'_u) \cdot R'_w|$ has the following leading term:

$$|(R'_t \times R'_u) \cdot R'_w| = \left| \frac{1}{2} f''_{uu}(t,0) \cdot \frac{u^2 \cdot \kappa_n(t)}{\sqrt{\kappa_n^2(t) + \tau_g^2(t)}} \right| + O(u^3) \quad . \tag{36}$$

The second derivative $f''_{uu}(t,0)$ is precisely the normal curvature of the surface S in the *direction of the ruling line* of the Cartan ribbon at $\gamma(t)$. It can thence be expressed by the curvature function values $H(t)$, $K(t)$, $\kappa_n(t)$ and $\tau_g(t)$ at $\gamma(t)$ along γ:

$$f''_{uu}(t,0) = \frac{\kappa_n}{\kappa_n^2 + \tau_g^2} \left(\tau_g^2 - \kappa_n^2 + 2H\kappa_n - 2\tau_g \sqrt{2H\kappa_n - K - \kappa_n^2} \right) \quad . \tag{37}$$

Insertion into (36) then gives:

$$
\begin{aligned}
\mathcal{M}(\gamma,\varepsilon) &= \int_J \int_{-\varepsilon}^{\varepsilon} \int_0^1 |(R'_t \times R'_u)\cdot R'_w| \; dt\, du\, dw \\
&= \int_J \int_{-\varepsilon}^{\varepsilon} \left| \frac{1}{2} \cdot \frac{u^2 \cdot \kappa_n^2}{\left(\kappa_n^2 + \tau_g^2\right)^{3/2}} \left(\tau_g^2 - \kappa_n^2 + 2H\kappa_n - 2\tau_g\sqrt{2H\kappa_n - K - \kappa_n^2}\right)\right| + O(u^3)\, dt\, du \\
&= \frac{1}{3}\varepsilon^3 \cdot \int_J \left| \frac{\kappa_n^2}{\left(\kappa_n^2 + \tau_g^2\right)^{3/2}} \left(\tau_g^2 - \kappa_n^2 + 2H\kappa_n - 2\tau_g\sqrt{2H\kappa_n - K - \kappa_n^2}\right)\right| \, dt + O(\varepsilon^4) \\
&= \frac{1}{3}\varepsilon^3 \cdot \int_J F(H(t), K(t), \kappa_n(t), \tau_g(t))\, dt + O(\varepsilon^4) \quad .
\end{aligned}
$$

□

Corollary 3.10. *Suppose that the center curve γ is a line of curvature on the surface S – as is the case for all the chosen center curves on the ellipsoid considered in section 6 below. Then the geodesic torsion of γ vanishes identically and the corresponding local measure of goodness of the Cartan ribbon along γ reduces to:*

$$
\mathcal{M}(\gamma,\varepsilon) = \frac{1}{3}\varepsilon^3 \cdot \int_J |\kappa_n(h(t))| \; dt + O(\varepsilon^4) \quad , \tag{38}
$$

where $\kappa_n(h(t))$ denotes the normal curvature of S at $\gamma(t)$ in the direction of $h(t)$, which is orthogonal to $\gamma'(t)$.

Proof. This follows directly from equation (36) and the fact that in this case we have $f''_{uu}(t,0) = \kappa_n(h(t))$. □

Another consequence of theorem 3.9 is the following result, which is not surprising, since we are approximating the surface S with flat Cartan ribbons:

Corollary 3.11. *Suppose that the Gaussian curvature K of S vanishes identically along γ. Then*

$$
\mathcal{M}(\gamma,\varepsilon) = O(\varepsilon^4) \quad . \tag{39}
$$

Proof. This follows readily by inserting the following ingredients into the formula (34):

$$
\begin{aligned}
K(t) &= 0\,, \\
H(t) &= \kappa_1(t)\,, \\
\kappa_2(t) &= 0\,, \\
\tau_g(t) &= \kappa_1(t)\cos(\theta(t))\sin(\theta(t))\,, \\
\kappa_n(t) &= \kappa_1(t)\cos^2(\theta(t))\,,
\end{aligned}
$$

where $\theta(t)$ denotes the angle between $\gamma'(t)$ and the principal direction of curvature for S at $\gamma(t)$ corresponding to the principal curvature $\kappa_1(t)$. □

Remark 3.12. *Although theorem 3.9 is but an initial step towards a global measure of goodness for the total number of individual Cartan ribbons (that are in use for the overall approximation of a given full surface), it may still be possible and reasonable to apply the formula* (34) *– or a proper refinement of it – for each* *ribbon and then simply sum the values of goodness over the number of ribbons. Naturally, the u-domain of integration should then not just be $[-\varepsilon, \varepsilon]$ but rather the full width-interval $[-w_-(t), w_+(t)]$ along the respective ribbons. Moreover, good single ribbon approximations (and their higher dimensional analogues) represent an interesting alternative basis and tool for principal geodesic analysis, and for polynomial regression in general, on surfaces and in Riemannian manifolds, see* [23] *and* [24]. *In particular, in that setting the notion of Riemannian polynomials have also been studied via rolling maps, see* [25] *and* [26] *– much in the same vein as we have employed the concept of rolling in the present work.*

The local cut-off procedure for neighboring ribbons. We consider two neighboring center curves γ^1 and γ^2 for two neighboring Cartan ribbons and prove the existence of their intersection curve, that eventually constitute the wedge (or cut-off) curve in space 'between' the two center curves, see the examples in sections 5 and 6. The wedge thereby defines the actual width functions w_-^2 and w_+^1, that are used for the final ribbonization of the surface S. In this setting w_+^1 is to be thought of as the cut off function for γ^1 in the direction towards γ^2, and w_1^2 is the cut off function for γ^2 in the (opposite) direction from γ^2 towards γ^1.

Proposition 3.13. *The wedges are well-defined for each pair of neighboring Cartan ribbons, i.e. the cut-off functions exist, provided the corresponding center curves are pairwise sufficiently close to each other.*

Proof. We sketch the proof as follows. Suppose that r_1 is the ruling line at some point p on γ^1. We must show that (for close-by neighboring center curves) there is a corresponding ruling line r_2 at some point of γ^2 so that the two rulings intersect in a (cut-off) point, i.e. so that w_+^1 and w_-^2 exist. Obviously, this does not necessarily work for center curves that are far apart from each other, so we need that the center curves are sufficiently close.

We may assume that the two center curves are neighboring coordinate curves in a special local parametrization of a tubular neighborhood around γ^1. Specifically, without lack of generality, we parametrize the neighborhood by a smooth vector function ρ with parameters t and v such that the following properties are satisfied: $\rho(t, 0) = \gamma^1(t)$; $\rho(t, \varepsilon) = \gamma^2(t)$; every t-coordinate curve has nonvanishing normal curvature: $\kappa_n(\rho_t'(t, v)) \neq 0$; and $\rho_v'(t_0, v)$ is in the direction of the ruling line of the Cartan ribbon along the curve $\rho(t, v)$ at the point $\rho(t_0, v)$ for all $v \in [0, \varepsilon]$.

This latter condition means that the curve $q_{t_0}(v) = \rho(t_0, v)$, $v \in [0, \varepsilon]$ has tangent lines that are ruling lines of the respective Cartan ribbons along the center curves $\rho(t, v)$ for each v in the said interval.

If the curve q_{t_0} has nonzero curvature at $v = 0$ (and possibly also nonzero torsion there), then an intersection argument in the ambient space shows that there exists a ruling line of the Cartan ribbon at some point $\rho(t_2, \varepsilon)$ along the center curve $\rho(t, \varepsilon)$ close to $\rho(t_0, \varepsilon)$, i.e. for t_2 close to t_0, which intersects the ruling line r_1 based at $p = \rho(t_0, 0)$ – provided ε is sufficiently small. If the torsion of the curve q_{t_0} vanishes in the interval $v \in [0, \varepsilon]$ so that it is planar in that interval, then $t_2 = t_0$ and the intersection takes place in that plane.

The same argument holds if q_{t_0} has zero curvature at $v = 0$ but, say, has positive curvature for $v \in]\delta, \varepsilon]$. Moreover, if q_{t_0} has zero curvature in an interval, $v \in [0, \varepsilon[$, then q_{t_0} is a straight line in that interval and every point on the ruler from p is also a point on a ruler for the ribbon with center curve $\eta(t, v_0)$ for any v_0 in that interval, and the corresponding cut-off value for w^1_+ can be chosen to be any value in $]0, \varepsilon[$. $\square$

4. Gauss Bonnet inspection

We consider a finite (piecewise smooth) ribbonization $\mathcal{R} = \cup_i^R \mathcal{R}_i$, $R = \#\mathcal{R}$, of S all of whose Cartan surface ribbons $\mathcal{R}_i$, $i = 1, \dots, R$, are closed in the sense that they are based on closed smooth center curves on S as in figures 2 and 4 below. Let $\mathcal{W} = \cup_i \mathcal{W}_i$ denote the system of (piecewise smooth) wedge curves stemming from the ribbonization $\mathcal{R}$ and let $\widehat{\mathcal{W}}$ denote the corresponding planar wedge curve system of the Cartan planar ribbons $\widehat{\mathcal{R}}$. The end (cut-)curves of the planar ribbons – that are typically needed in order to obtain the planar representation of the ribbons – are not considered part of $\widehat{\mathcal{W}}$.

We now apply the Gauss Bonnet theorem to surfaces which are ribbonized by such circular ribbons.

The system of wedge curves $\mathcal{W}$ consists of curves with possible branch-points, where three or more ribbons come together, and with possible end-points, where one ribbon is locally bent around the wedge (and is thus in contact with itself), as in the top and bottom ribbon on the ellipsoid in Fig. 4 below.

We may assume without lack of generality that the branch points and end points are all isolated and regular in the sense that the wedge curves in a neighbourhood of such points can be mapped diffeomorphically to a corresponding star configuration in $\mathbb{R}^3$ with a number of straight line segments issuing from a common vertex. The branch-points and end-points are called *vertices* of the ribbonization $\mathcal{R}$. The vertex set is denoted by $\mathcal{P}$ and the number of vertices by $P = \#\mathcal{P}$. The number of segments issuing from a given vertex p_k in the vertex set $\mathcal{P}$ is called the *degree*, $d_k = d(p_k)$ of the vertex. If a ribbon has an isolated cone point then this is also a vertex, and – in accordance with the above definition – we count its degree as 0.

Theorem 4.1. *The Euler characteristic, $\chi(\mathcal{R})$, of a ribbonization $\mathcal{R}$ is*

$$\chi(\mathcal{R}) = \frac{1}{2}\sum_{k=1}^{P}(2 - d_k) \quad . \tag{40}$$

Proof. The total curvature contributions for the Gauss–Bonnet theorem can be divided into three parts:

a) *Surface contributions:* the surface integral of the Gauss curvature K,

$$C_{\mathcal{R}\setminus\mathcal{W}\cup\mathcal{P}} = \int_{\mathcal{R}\setminus\mathcal{W}\cup\mathcal{P}} K d\mu = 0 \quad . \tag{41}$$

b) *Wedge contributions:* The integral of the geodesic curvature along the edges of the Cartan ribbons excluding the vertex points,

$$C_{\mathcal{W}\setminus\mathcal{P}} = \sum_{q=1}^{R} \int_{\mathcal{W}_q\setminus\mathcal{P}_q} \kappa^{\mathcal{W}_q}(s)\, ds \quad . \tag{42}$$

c) *Vertex contributions:* sum of the angular deficit (angular defect) at the vertices, i.e. 2π minus the sum of the inner angles $\beta(j,k)$ at the vertices. The inner angles are replaced by the corresponding outer angles $\alpha(j,k)$ as $\alpha = \pi - \beta$ where $\alpha \in [-\pi, \pi]$ and $\beta \in [0, 2\pi]$,

$$
\begin{aligned}
C_{\mathcal{P}} &= \sum_{j=1}^{P} \left(2\pi - \sum_{k=1}^{d_k} \beta(j,k) \right) \\
&= \sum_{k=1}^{P} \left(2\pi - \sum_{j=1}^{d_k} (\pi - \alpha(j,k)) \right) \qquad (43) \\
&= \sum_{k=1}^{P} (2\pi - \pi d_k) + \sum_{k=1}^{P} \sum_{j=1}^{d_k} \alpha(j,k) \quad .
\end{aligned}
$$

Summarising: Adding these contributions together we find:

$$
2\pi \cdot \chi(\mathcal{R}) = \sum_{q=1}^{R} \int_{\mathcal{W}_q \setminus \mathcal{P}_q} \kappa^{\mathcal{W}_q}(s)\, ds + \sum_{k=1}^{P} \sum_{j=1}^{d_k} \alpha(j,k) + \sum_{k=1}^{P} (2\pi - \pi d_k) \quad . \tag{44}
$$

By a permutation of the outer angles in the second term one can group them according to the ribbon wedge curves they appear on. This is possible because each of the kinks on the ribbons is encountered precisely once in the summation. Further, as the ribbons are closed it follows that their wedge integral and the corresponding sum of outer angles together cancels to zero. Hence one is left with the equality:

$$
\chi = \frac{1}{2} \sum_{k=1}^{P} (2 - d_k) \quad . \tag{45}
$$

□

Remark 4.2. *As mentioned, the set of vertices, $\mathcal{P}$, is a feature of the three-dimensional mesh of wedge curves. Wedge curves from two, most commonly distinct, ribbons follow each other until a vertex point, where, e.g. three ribbons come together. We summarise the different vertex characters with Table 1.*

d_k	Vertex character	Classification
0	*Fully circumscribed by one ribbon*	*Cone point*
1	*Half circumscribed by a ribbon*	*Wedge end point*
2	*Two ribbons meet at the point*	*Zero contributing vertex*
> 2	*$n > 2$ ribbons meet*	*Conventional vertex*

Table 1. A characterisation of vertices.

The ribbon formula in Theorem 4.1 is valid for orientable as well as non-orientable surfaces. To see this we only need to show that the formula does not change whether the ribbons are regular closed ribbons or Möbius strip-ribbons. This follows as a consequence of lemma 4.3 below.

Lemma 4.3. *A conventional cylindrical closed ribbon (without vertices), and a Möbius strip-ribbon both contribute zero to the total curvature integral.*

Proof. It follows simply by cutting the ribbons along a ruler. In this case, the ribbons can be fully flattened and has a total curvature contribution of 2π which is equal to the sum of the four artificial angles introduced by the cutting along the ruler. The difference between a ribbon that is orientable and one that is not consists of a simple permutation of the four inner angles of the cut. □

For the explicit extension of theorem 4.1 to Cartan ribbonizations that include ribbons with open center curves, it is sufficient to count the number N_O of such ribbons, and equation 40 becomes:

$$\chi = \frac{1}{2}\sum_{k=1}^{P}(2 - d_k) + N_O \quad . \tag{46}$$

Remark 4.4. *A necessary and sufficient criterion for the correct representation of the topology of the surface S by a given ribbonization is the following: For each ribbon there exists a homeomorphism which maps the ribbon to a domain on the surface such that*

(1) *The contact structure (edges and vertices) between the individual ribbons is preserved*
(2) *The full surface S is covered precisely once by the images of the ribbons.*

For ribbonizations with sufficiently narrow ribbons, i.e. with small cut-off functions w_- and w_+, such homeomorphisms can for example be obtained via normal projection (along the orthogonal lines to S) of the ribbons into the surface.

From ribbon inspection to the Euler polyhedral formula. We consider a polyhedron Q and apply the conventional notation, i.e. F, E and V denote the number of faces, of edges, and of vertices, respectively, of the *polyhedron*. To apply the ribbon formula (40) we need to cover the polyhedron with closed ribbons. One can cover each one of the F faces by a closed ribbon with a (flat) vertex covering the intrinsic part of the face polygon. With this choice there are then F new such virtual vertices, all with degree zero. We therefore have the total number of ribbon vertices

$$P = V + F \tag{47}$$

and

$$\sum_{k=1}^{P} d_k = 2 \cdot E \quad . \tag{48}$$

Hence we recover the well known polyhedron formula from the ribbon formula:

$$\chi(Q) = \frac{1}{2}\sum_{k=1}^{V}(2 - d_k) = \frac{2(V + F) - 2E}{2} = V - E + F. \tag{49}$$

5. An unknot-based Cartan ribbonized torus

This example is concerned with the ribbonization of the torus

$$\mathcal{T}^2 : \ \sigma(u,v) = \big((2+\cos(u))\cdot\cos(v)\,,\ (2+\cos(u))\cdot\sin(v)\,,\ \sin(u)\big)\,,\ (u,v)\in\mathbb{R}^2 \tag{50}$$

using the following two closed curves as center curves (see Fig. 1):

$$\begin{aligned} \gamma_1(t) &= \sigma\,(3\cdot t\,,\ t) \quad , \quad t\in[-\pi,\pi] \\ \gamma_2(t) &= \sigma\left(3\cdot t\,,\ t+\frac{\pi}{3}\right) \quad , \quad t\in[-\pi,\pi] \quad . \end{aligned} \tag{51}$$

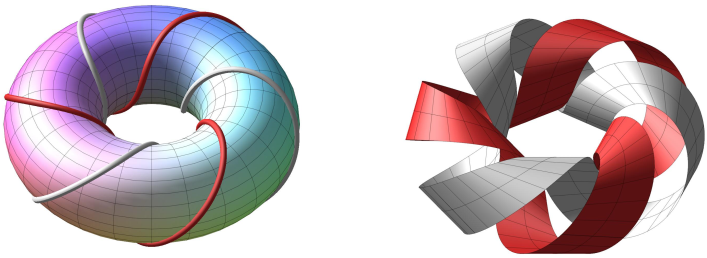

Figure 1. Two $(3,1)$-unknots on a torus that are used as center curves for the beginning of a Cartan ribbonization of the torus. See Fig. 2 with the corresponding ribbons, extended and cut-off.

The corresponding two Cartan surface ribbons are then constructed (with constant and equal width functions) along the two curves, using the parametrization recipe in (21). They are displayed on the right in Fig. 1. The ribbons are then widened in $\mathbb{R}^3$ in the direction of $\pm\omega$ until intersection with their respective neighbour ribbons.

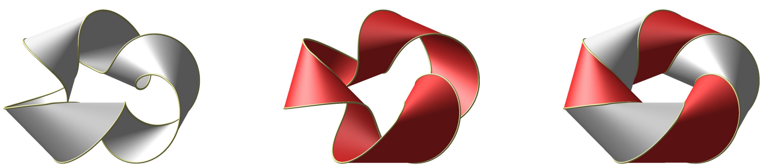

Figure 2. Ribbonization of the torus along two $(3,1)$-unknots with the correct cut-off width functions.

In the present example the planar ribbons are constructed via the planar center curves $\tilde{\gamma}$ from (25) using the geodesic curvature function from the curves (51) on the torus, see figure 3.

The intersection width functions are obtained numerically by solving the intersection equation for each value of t along the center curves, see Fig. 2. Once the cut-off widths $w_\pm$ of the Cartan surface ribbons have been determined, the corresponding Cartan planar ribbons (with the same width-functions $w_\pm(t)$) are

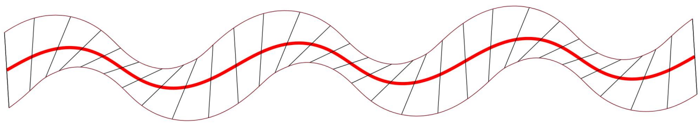

FIGURE 3. The geometry of one of the two identical planar ribbons used for the covering of the torus in Fig. 2.

finally constructed from the planar center curve with the same geodesic curvature as the original center curve on the surface. In this particular case both Cartan planar ribbons are identical – one of them is displayed in Fig. 3.

Inspection of the ribbonized torus. The number of vertices of the above ribbonization is 0 and hence according to equation (45) we get immediately Euler characteristic $\chi = 0$ for the torus.

6. Curvature line based ribbonizations of an ellipsoid

A curvature line parametrization of the ellipsoid with half axes $\sqrt{a} > \sqrt{b} > \sqrt{c} > 0$ is obtained as follows, see [27] and [9, Example 7.4]:

(52)
$$\sigma(u,v) = \Big(\pm\sqrt{\frac{a(a-u)(a-v)}{(a-b)(a-c)}}, \pm\sqrt{\frac{b(b-u)(b-v)}{(b-a)(b-c)}}, \pm\sqrt{\frac{c(c-u)(c-v)}{(c-a)(c-b)}} \Big),$$

where $u \in (b,a)$ and $v \in (c,b)$. This particular parametrization of the ellipsoid is shown in the leftmost display in Fig. 4. As shown on the display the coordinate (curvature) lines of this parametrization extend smoothly from one octant to a neighbouring octant except at the 4 umbilical points on the ellipsoid corresponding to parameter values $u \to b$ and $v \to b$.

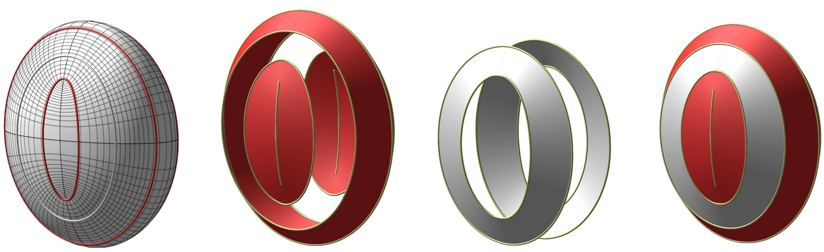

FIGURE 4. Ribbonization of the ellipsoid with half axes $\sqrt{5}$, 2, and 1 corresponding to the parameters $(a,b,c) = (5,4,1)$ in the representation (52). The ribbonization is built from 6 Cartan surface ribbons along the indicated line-of-curvature center-lines which intersect the horizontal "equator" at equi-distributed points.

Such curvature line ribbonizations are interesting, partly because they give nontrivial illustrations of the simple measure of goodness established in corollary

3.10, and partly because they also clearly highlights the significant umbilical points. The umbilics on the ellipsoid considered here correspond to the four endpoints of the wedge segments that appear on the top cap and on the bottom cap – both visible in the second display from the left in figure 4.

Inspection of the ellipsoid. The ellipsoid has 4 vertices – corresponding to the 4 umbilical points – each of degree one, $d_k = 1$, and each therefore contributing one-half to the Euler characteristic, see equation (40).

7. Comparison with classical topological inspections

As illustrated above, the topology of the surface can be read off from a ribbonization – in fact often in an easier way than from a triangulation. In this section we will briefly compare the above inspection with the methods of Morse and Poincaré-Hopf based on inspections of Morse height-functions and their corresponding vector fields, respectively.

Consider a Morse height-function f on a surface S and choose center curves for a ribbonization among level curves of f. Since the saddle points of f are isolated the center curves can be chosen to be arbitrarily close and yet with tangents avoiding asymptotic directions, so that such ribbonizations exist and have the same topology as the surface. Moreover, as a third perspective, the gradient of f on S represents a vector field whose indices also count its topology.

Based on a Morse height-function these three topological inspections are all based on countings of minima, maxima, and saddle points. Clearly, the final summations give the same result when applying Table 2 below.

	Ribbon (d_k)	Morse (γ)	Vector field (I)
Minimum	0	0	1
Saddle point	4	1	-1
Maximum	0	2	1
χ	$\frac{1}{2}\sum_{k=1}^{n_v}(2-d_k)$	$\sum_{\gamma=0}^{\gamma=2}(-1)^\gamma\, n_\gamma$	$\sum_{k=1}^{n_z} I_k$

Table 2. Listing the relationship between degrees of vertices, d_k, Morse indices γ, and the indices, I, of a vector field for smooth two-dimensional manifolds. Also compared are the corresponding three topological inspections for the Euler characteristic: the ribbon inspection, based on vertex counting; the Morse index formula, based on critical points of Morse functions [28]; and the Poincaré-Hopf formula, based on the counting of types of zeros of a vector field [29].

In the case of a torus with its classical Morse height function, see [30, Diagram 1 p. 1], the corresponding ribbonizations all have one minimum, one maximum (both with degree $d_k = 0$) and two saddle points (with degrees $d_k = 4$), so that the sum is $\frac{1}{2}\sum_{k=1}^{n_v}(2-d_k) = 0$, as expected. An interesting Morse height function for the non-orientable Boy's model of $\mathbb{R}P^2$ in $\mathbb{R}^3$, that may likewise be used as center curves for a ribbonization, is presented by U. Pinkall in [31, Chapter 6, pp. 63–67, figures 6.7 and 6.8]. This particular ribbonization has 4 vertices of degree $d_k = 0$, and 3 vertices of degree $d_k = 4$, so that $\chi = 1$.

8. Conclusions

In this paper we recover the conditions for the existence of proper rollings of one surface on another [3, 4, 5, 6] – in particular the condition that the two contact curves, that are generated from the rolling, have identical geodesic curvature. This follows from defining the standard rollings as rigid motions in $\mathbb{R}^3$ that are conditioned partly via their instantaneous rotation vectors and partly via the obvious condition of contact between the mentioned track curves on the respective surfaces, i.e. common speed of the contact point along the tracks and common tangent planes at the instantaneous point of contact.

Surfaces are then approximated by a mesh of ribbons. Rolling a surface on a plane and using the Cartan developments of curves allow us to construct developable ribbons that have common tangent planes everywhere along the curve of contact on the surface. In this way we may approximate the surface not just by one such developable surface but by a full set of ribbons. In short, the surface is ribbonized by flat ribbons which have center-curve contact with the surface. This is a clear difference in comparison with the much used method of triangulations, which typically only give discrete point-contact with the surface. In the same way as for triangulations, defining a measure of "goodness" of a Cartan ribbonization is dependent on the actual application. Different methods for designing surfaces by developable patches within a desired global error bound have been developed in [32, 33, 34, 35]. For Cartan ribbonizations, this is an interesting problem, which we have addressed by introducing a local measure of goodness for the approximation of the surface along a single ribbon.

Concerning the global structure of the approximations, we present a particularly simple topological inspection of the ribbonized surfaces, which gives the Euler characteristic of the ribbonization – and thence also of the surface, if the ribbonization is fine enough. The ensuing topological formula for the Gauss-Bonnet theorem involves only the vertices of the ribbonization and their degrees. This complements the classical inspections of topology stemming from Morse theory and from the Poincaré-Hopf formula, which also amount to summing over critical point indices. If we organise the ribbonization of a given surface according to level curves of a Morse height function, then we obtain the direct correspondence shown in Table 2.

The intriguing relations between the kinematics of rolling and the geometry of developable surfaces clearly carries many more assets for future work than what we cover in the present paper. As indicated above, already the study of ribbonizations could well pave new ways for refined analyses of physical, geometrical, and topological properties of surfaces. Not to mention the potentials of their higher dimensional analogues. Possible practical applications are manifold and appear in such diverse fields as robotics, architecture, design, shape analysis, and modern engineering. See for example the following works on rolling spherical robots [36], roof panelling [37], statistical geometric regression analysis [23, 24], and the manufacturing of clothes [38].

References

[1] Schumaker LL.
1993 Triangulations in CAGD.
IEEE Computer Graphics and Applications **13**, 47-52.

[2] Lawrence S.
2011 Developable surfaces: Their history and application.
Nexus Network Journal **13**, 701-714.

[3] Nomizu K.
1978 Kinematics and differential geometry of submanifolds.
Tôhoku Math. Journ. **30**, 623-637.

[4] Kobayashi S, Nomizu K.
1963 *Foundations of Differential Geometry I*,
Interscience Publishers, New York.

[5] Tunçer Y, Sağel MK, Yayli Y.
2007 Homothetic motion of submanifolds on the plane in $\mathbb{E}^3$.
Journal of Dynamical Systems & Geometric theories **5**, 57-64.

[6] Cui L, Dai JS.
2010 A Darboux-frame-based formulation of spin-rolling motion of rigid objects with point contact.
IEEE Transactions on Robotics **26**, 383-388.

[7] Molina MG, Grong E.
2014 Geometric conditions for the existence of a rolling without twisting or slipping.
Communications on Pure and Applied Analysis **13**, 435-452.

[8] Izumiya S, Otani S.
2015 Flat Approximations of Surfaces Along Curves.
Demonstratio Mathematica **48**, 217-241.

[9] Hananoi S, Izumiya S.
2017 Normal developable surfaces of surfaces along curves.
Proceedings of the Royal Society of Edinburgh Section A: Mathematics pp. 1-27.

[10] Raffaelli M, Bohr J, Markvorsen S.
2016 Sculpturing surfaces with Cartan ribbons.
In Proceedings of Bridges 2016: Mathematics, Music, Art, Architecture, Education, Culture. pp. 457-460. Tessellations Publishing. (Bridges Conference Proceedings).

[11] Cui L, Dai JS.
2015 From sliding-rolling loci to instantaneous kinematics: An adjoint approach.
Mechanism and Machine Theory **85**, 161-171.

[12] Chitour Y, Molina MG, Kokkonen P.
2015 Symmetries of the rolling model.
Mathematische Zeitschrift. **281**, 783-805.

[13] Tunçer Y, Ekmekci N.
2010 A study on ruled surface in euclidean 3-space.
Journal of Dynamical Systems & Geometric theories **8**, 49-57.

[14] Chitour Y, Molina MG, Kokkonen P.
2014 The rolling problem: overview and challenges in Geometric Control Theory and Sub-Riemannian Geometry.
Editors: G. Stefani, J.P. Gauthier, M. Sigalotti, U. Boscain, A. Saryehev, Springer International Publishing, 2014. 103-122.

[15] Krakowski KA, Leite FS.
2016 Geometry of the rolling ellipsoid.
Kybernetika **52**, 209-223.

[16] Sharp J.
2005 D-forms and Developable Surfaces.
In Bridges 2005: Mathematical Connections between Art, Music and Science 503-510.

[17] Wills T.
2006 D-Forms: 3D forms from two 2D sheets.
In Bridges 2006: Mathematical Connections in Art, Music, and Science 503-510.
[18] Orduño RR, Winard N, Bierwagen S, Shell D, Kalantar N, Borhani A, Akleman EA.
2016 A Mathematical Approach to Obtain Isoperimetric Shapes for D-Form Construction. In Proceedings of Bridges 2016: Mathematics, Music, Art, Architecture, Education, Culture 277-284. Tessellations Publishing.
[19] Gray A, Abbena E, Salomon S.
2006 *Third Edition: Modern differential geometry of curves and surfaces with Mathematica®*, ISBN: 978-0-58488-488-4. CRC Press, Boca Raton, Florida.
[20] do Carmo MP.
1976 *Differential geometry of curves and surfaces*, ISBN: 978-0132125895. Prentice Hall, Englewods Cliffs, New Jersey.
[21] Izumiya S, Saji K, Takeuchi N.
2017 Flat surfaces along cuspidal edges.
Journal of singularities **16**, 73-100.
[22] Singer IM, Thorpe JA.
1967 *Lecture notes on elementary topology and geometry.*
ISBN: 0-387-90202-3. Springer, New York, NY.
[23] Fletcher PT, Lu CL, Pizer SA, and Joshi S.
2004 Principal geodesic analysis for the study of nonlinear statistics of shape.
Ieee Transactions on Medical Imaging **23**, 995-1005.
[24] Hinkle J, Muralidharan P, Fletcher, PT, Joshi S.
2012 Polynomial Regression on Riemannian Manifolds.
Lecture Notes in Computer Science **7574**, 1-14.
ISBN: 9783642337116. Springer, Berlin.
[25] Jupp PE, Kent JT.
1987 Fitting smooth paths to spherical data.
Journal of the Royal Statistical Society Series C-applied Statistics **36**, 43-46.
ISSN: 14679876, 00359254.
[26] Leite FS, Krakowski KA.
2015 Covariant differentiation under rolling maps.
Preprint Number 0822, Universidade de Coimbra.
[27] Sotomayor J, Garcia R.
2008 Lines of Curvature on Surfaces, Historical Comments and Recent Developments.
São Paulo Journal of Mathematical Sciences **2**, 99-143.
[28] Ni X, Garland M, Hart JC.
2004 Fair Morse functions for extracting the topological structure of a surface mesh.
ACM Transactions on Graphics (TOG) **23**, 613-622.
[29] Milnor JW.
1965 *Topology from the differentiable viewpoint.*
ISBN: 0-691-04833-9. The University Press of Virginia, Charlottesville.
[30] Milnor JW.
1965 *Morse Theory.*
Annals of Mathematics Studies 51.
Princeton University Press.
[31] Barth W, et al.
1986 *Mathematical models.*
ISBN: 3-528-08991-1. Friedr. Vieweg & Sohn, Braunschweig.
[32] Liu Y-J, Lai Y-K, Hu S-M.
2007 Developable strip approximation of parametric surfaces with global error bounds.
Pacific Graphics 2007: 15th Pacific Conference on Computer Graphics and Applications, pp. 441-444.

[33] Liu Y-J, Lai Y-K, Hu S.
2009 Stripification of Free-Form Surfaces With Global Error Bounds for Developable Approximation.
IEEE Transactions on Automation Science and Engineering **6** 700-709.

[34] Tang C, Bo P, Wallner J, Pottmann H.
2016 Interactive Design of Developable Surfaces.
Acm Transactions on Graphics **35** 1-12.

[35] Rabinovich M, Hoffmann T, Sorkine-Hornung O.
2018 Discrete geodesic nets for modeling developable surfaces.
ACM Transactions on Graphics **37** 16:1-16:17.

[36] Bai Y, Svinin M, Yamamoto M.
2015 Motion planning for a pendulum-driven rolling robot tracing spherical contact curves.
Ieee International Conference on Intelligent Robots and Systems, 4053-4058.
ISBN: 9781479999941, 9781479999934. Institute of Electrical and Electronics Engineers Inc.

[37] Mehrtens P, Schneider M.
2012 Bahn frei für die Architektur – Approximation von Freiform-Flächen durch abwickelbare Streifen.
Stahlbau, **81**, Heft 12, 931-934.

[38] Rose K, Sheffer A, Wither J, Cani M P, Thibert B.
2007 Developable Surfaces from Arbitrary Sketched Boundaries.
Eurographics Symposium on Geometry Processing (2007).
The Eurographics Association 2007.

DTU Nanotech, Technical University of Denmark, 2800 Kongens Lyngby, Denmark

E-mail address: jabo@nanotech.dtu.dk

DTU Compute, Technical University of Denmark, 2800 Kongens Lyngby, Denmark

E-mail address: stema@dtu.dk

DTU Compute, Technical University of Denmark, 2800 Kongens Lyngby, Denmark

E-mail address: matraf@dtu.dk

NEWSON'S CHALLENGE AND THE VOLUME OF CERTAIN CONVEX HULLS

JAKOB BOHR, STEEN MARKVORSEN, AND MATTEO RAFFAELLI

ABSTRACT. We comment on a challenge raised by Newson in the Annals more than a century ago and present an expression for the volume of the convex hull of a convex closed space curve with four vertex points.

1. INTRODUCTION

In 1899 H.B. Newson published an exposition concerning the volume of a polyhedron with n faces [9]. In the afterword he called on the readers of Annals of Mathematics to find a formula for the volume of a closed surface, in analogy to the formula for the area of a closed planar curve, c,

$$A = \frac{1}{2}\int_c (xdy - ydx)\,. \tag{1}$$

Often the well known Equation (1) is explained on the basis of Green's formula. Newson derived the area formula based on the limiting case of a many-sided polygon, and then hinted that a similar approach might provide insight into the volume of a closed surface. Perhaps he had in mind something like the divergence theorem. Here, we will follow a different path, namely one of integrating along closed space curves. First we restate the area derivation of Newson in a slightly disguised form.

For a closed, and simple curve, $\alpha \in \mathbb{R}^3$, that is constricted to a 2-plane containing the origin, we can write the area of the enclosed domain Ω, $\alpha = \partial\Omega$, as

$$A(\Omega) = \frac{1}{2}\left| \oint_\alpha \dot{r}(s) \times r(s)\, ds \right|. \tag{2}$$

where $r(s)$ is a unit speed parameterization for α. This result can be obtained by taking the limiting case of an n-polygon with vertices all constricted to the same 2-plane in $\mathbb{R}^3$. With vertices at $\{r_i\}$, $i \in \{1, \dots, n\}$, the area A_P becomes,

$$A_P = \frac{1}{2}\left| \sum_{i=1}^{n} r_i \times r_{i+1 \bmod n} \right|. \tag{3}$$

In the following we outline how a generalization of Equation (2) leads to a simple expression for the volume of the convex hull of a convex space curve with 4 vertices:

Theorem 1. *Let γ be a unit speed convex space curve parameterized by $r(s)$, and assume that γ has 4 vertices. Then the volume of the convex hull is:*

$$V(\mathrm{Conv}(\gamma)) = \frac{1}{24}\oint_\gamma \oint_\gamma |[\dot{r}(s_1)\dot{r}(s_2)(r(s_1) - r(s_2))]|\, ds_1 ds_2\,. \tag{4}$$

2. Proof of the theorem

A space curve is by definition convex if every point is an extreme point. Then the convex hull of a convex space curve consists of two developable sheets that meet at a seam curve (the given space curve). We refer to [2, 6, 7, 8] for details.

Proof. We investigate cases where one can parameterize the hull, $\mathrm{Conv}(\gamma)$, with $s_1 \in [0, L]$, $s_2 \in [0, L]$ and $u \in [0, 1]$ as follows:

$$r(s_1, s_2, u) = \gamma(s_1) + u(\gamma(s_2) - \gamma(s1))\,. \tag{5}$$

This parameterization utilizes the 4 vertex requirement, which secures that all points of $\mathrm{Conv}(\gamma)$ can be written as a convex sum of just two extreme points. If γ has precisely four points of zero torsion and no points of zero curvature then γ has no tri-tangential supporting planes and the convex hull of γ has no polygonal domains. This result follows straightforwardly from generalized versions of the four-vertex theorem [11, Corollary 1]: $V + 2K + 2d \geq 4 + P$ where V is the number of vertex points (with zero torsion), K is the number of zero curvature points, d the number of external segments, and P the number of support polygons. Note, that $d = 0$ when the curve lies on its convex hull [4]. If the boundary of the convex hull contained a planar triangle spanned by the vertices V_1 ,V_3 and V_5, then the point $(V_1 + V_3 + V_5)/3$ cannot be described by the assumed parametrization of the hull. For more details on tri-tangential planes, see ref. [4, 3, 1, 10].

Now we resume to the volume formula. First we observe that

$$\begin{aligned} \frac{\partial r}{\partial s_1} &= (1-u)e(s_1) \\ \frac{\partial r}{\partial s_2} &= ue(s_2) \\ \frac{\partial r}{\partial u} &= \gamma(s_2) - \gamma(s_1)\,. \end{aligned} \tag{6}$$

so that the Jacobian determinant needed for the volume element becomes:

$$\left[\frac{\partial r}{\partial s_1}, \frac{\partial r}{\partial s_2}, \frac{\partial r}{\partial u}\right] = (u - u^2)[e(s_1), e(s_2), \gamma(s_2) - \gamma(s_1)]\,. \tag{7}$$

Let m be the number of times a point of the internal hull is encountered by the chosen parameterization. Then

$$V = \frac{1}{6m} \oint_\gamma \oint_\gamma |[e(s_1), e(s_2), \gamma(s_2) - \gamma(s_1)]| \, ds_1 ds_2 \, . \tag{8}$$

where the factor 1/6 comes from integrating over the parameter u. Thus, in the spirit of polygonal discretisation, we are summing volumes of tetrahedra where two of the edges are infinitesimal. The connection to the challenge by Newson can be seen by investigating the case where the curve γ is approximated by smaller and smaller straight line segments. It forms a limiting case where γ is akin to a non-flat n-sided polygon. Two neighboring polygonal vertices $\{r_i, r_{i+1}\}$ forms a straight segment. With a double sum we can combine a straight segment from the first sum with a segment from the second sum, i.e. the four points $\{r_i, r_{i+1}, r_j, r_{j+1}\}$, which span an elongated tetrahedron, henceforth denoted by $T_{i,j}$. It has the signed volume:

$$V_{i,j} = \frac{1}{6}[r_{i+1} - r_i, r_j - r_i, r_{j+1} - r_i] = \frac{1}{6}[r_{i+1} - r_i, r_j - r_i, r_{j+1} - r_j] \tag{9}$$

Increasing, or decreasing, one of the two indices $\{i, j\}$ by a single step, one generates an adjacent tetrahedron, e.g. $T_{i+1,j}$, which shares a planar triangle with $T_{i,j}$. A pertinent question is then, are the interior of the two tetrahedra disjoint or not? In other words, are they in different halfspaces, or in the same halfspace, defined by the plane containing the shared triangle?

If

$$\text{sign}(V_{i,j}) = \text{sign}(V_{i+1,j}) \, , \tag{10}$$

then the interior of the two tetrahedra are disjoint, and the tetrahedra are sharing only the common planar triangle. However, if

$$\text{sign}(V_{i,j}) = -\text{sign}(V_{i+1,j}) \, . \tag{11}$$

then the shared triangle is on the boundary of the convex hull and the two tetrahedra are in the same halfspace. Therefore their volumes are not disjoint.

In general, a tetrahedron $T_{i,j}$ will fill some interior part of the hull that has contact with the boundary of the hull at the four coordinates $\{r_i, r_{i+1}, r_j, r_{j+1}\}$. Will all of the tiny tetrahedra together fill the convex hull? Yes, $T_{i,j}$ has four triangular faces of which each is shared with one of the following tetrahedra $\{T_{i+1,j}, T_{i-1,j}, T_{i,j+1}, T_{i,j-1}\}$. In other words, no tetrahedron $T_{k,l}$ will have a triangular face that are not met by a neighboring tetrahedron except for those on the boundary of the hull.

Upon increasing index j while keeping index i fixed, a face of the tetrahedron will reach the boundary of the hull. This occurs once at each of the two developable boundary sheets of the hull, thereby contributing a factor of two to m. An additional factor of two comes from from considering the index i, i.e. from interchanging i and j. Hence, we are left with $m = 4$.

□

3. Applications to D-forms

About a decade ago, the artist Tony Wills developed work on so-called D-forms, i.e. compact domains bounded by two developable sheets that are seamed together along their respective boundary curves [12]. For an introduction to D-forms, see also ref. [13]. We define a D-form as follows:

Definition 3.1. *A D-form is a compact domain in $\mathbb{R}^3$ whose boundary consists of two connected developable sheets, such that the boundaries of these two sheets are identified to form a closed space curve γ, the seam curve of the D-form.*

Observation 3.1. *The seam curve γ forms the extreme points of the D-form.*

Proof. Since the two sheets are developable all the extreme points must be on their boundary, i.e. points of the seam curve γ. Further, γ must be a closed space curve and every point of γ an extreme point. The latter follows from the definition of a D-form, as it requires exactly two developable sheets. □

Remark 3.1. *Some authors use a more extended definition of D-forms, e.g. one which allows for cone-points and/or creases on the developable sheets. In case of creases where the two sheets have merged and become a single sheet with a fold the resulting structure has been called a pita-form* [5].

Observation 3.2. *A D-form is the convex hull of its seam curve.*

Proof. It follows from the uniqueness of the convex hull and from the fact that the seam curve consists of the extreme points. □

In general the D-forms defined by Definition 3.1 can have boundary areas that are not just intrinsically flat but also extrinsically planar, e.g. planar triangles. A more demanding requirement is to limit the D-forms considered to those without any planar triangles (or other polygons) on the surface of the developable sheets. The volume of such a simple D-form is then given by Theorem 1.

4. Conclusion

A surprisingly simple formula for the volume of convex hulls of certain closed space curves is obtained. It involves a simple idea of using a Jacobian determinant for a choice of parameterization of the convex hull that is not one-to-one. The main observation is that the chosen parameterization covers all internal points of the hull exactly four times. Changes in orientation are handled by taking the absolute value of the volume element. The requirement that the internal points of the hull are covered an equal number of times by the parameterization is what limits the validity of the formula to closed space curves with only 4 vertex points.

The artistic expressions of D-forms by T. Wills call for a revisit to the study of convex hulls of closed space curves. John Sharp argues in his booklet that D-forms are beautiful objects [14]. Could it be because the human eye recognizes them as minimal convex sets? To try and answer this question goes beyond the scope of the current work, although it could form the basis for an interesting enquiry.

References

[1] E.L. Arima.
A Fabricius-Bjerre type invariant of space curves Ph.D. thesis, University of California, Davis (2005).

[2] M. Berger.
Geometry Revealed, DOI 10.1007/978-3-540-70997-8-7
Springer-Verlag Berlin Heidelberg 2010.

[3] T. Banchoff, T. Gaffney, and C. McCrory.
Counting tritangent planes of space curves Topology **24** (1985) 15-23.

[4] S.R. Costa, and M.D.C. Romero-Fuster.
Nowhere vanishing torsion closed curves always hide twice.
Geometriae Dedicata **66** (1997) 1-17.

[5] E.D. Damaine and G.N. Price.
Generalized D-forms have no spurious creases.
Discrete Comput. Geom. **43** (2010) 179-186.

[6] W. Fenchel.
On the differential geometry of closed space curves.
Bull. Amer. Math. Soc. **57** (1951) 44-54.

[7] B. Grünbaum. Convex Polytopes, ISBN 0-387-40409-0.
Springer-Verlag, New York 2003.

[8] V. Klee.
What is a convex set?
The American Mathematical Monthly **78** (1971) 616-631.

[9] H.B. Newson.
On the volume of a polyhedron.
Annals of Mathematics, Second Series **1** (1899) 108-110.

[10] K. Ranestad and B. Sturmfels.
On the convex hull of a space curve.
Advances in Geometry **12** (2012) 157-178.

[11] M.C. Romero-Fuster and V.D. Sedykh.
A lower estimate for the number of zero-torsion points of a space curve.
Contributions to Algebra and Geometry **38** (1997) 183-192.

[12] T. Wills.
D-Forms: 3D Forms from two 2D Sheets.
Proceedings of Bridges: Mathematical Connections in Art, Music, and Science, (2006) 503510.

[13] J. Sharp.
D-Forms and Developable Surfaces.
Proceedings of Bridges: Mathematical Connections in Art, Music and Science, (2005), pp. 121-128.

[14] J. Sharp.
D-Forms: surprising new 3D forms from flat curved shapes.
Tarquin Publishers, St. Albans, UK 2005.

DTU Nanotech, Technical University of Denmark, DK 2800 Kongens Lyngby, Denmark
E-mail address: jabo@nanotech.dtu.dk

DTU Compute, Technical University of Denmark, DK 2800 Kongens Lyngby, Denmark
E-mail address: stema@dtu.dk

DTU Compute, Technical University of Denmark, DK 2800 Kongens Lyngby, Denmark
E-mail address: matraf@dtu.dk

manuscripta math.

Irina Markina · Matteo Raffaelli

Flat approximations of hypersurfaces along curves

Received: 20 June 2018 / Accepted: 4 October 2018

Abstract. Given a smooth curve γ in some m-dimensional surface M in $\mathbb{R}^{m+1}$, we study existence and uniqueness of a flat surface H having the same field of normal vectors as M along γ, which we call a flat approximation of M along γ. In particular, the well-known characterisation of flat surfaces as torses (ruled surfaces with tangent plane stable along the rulings) allows us to give an explicit parametric construction of such approximation.

1. Introduction and main result

Developable, or flat, hypersurfaces in $\mathbb{R}^{m+1}$, where $m \geq 2$, are classical objects in Riemannian geometry. They are characterised by being foliated by open subsets of $(m-1)$-dimensional planes, called rulings, along which the tangent space remains stable [16, Theorem 1]. Here we are concerned with the problem of existence and uniqueness—as well as with the explicit construction—of flat approximations of hypersurfaces along curves. Let M^m be a (possibly curved) Euclidean hypersurface and γ a curve in M^m. A hypersurface H is called an *approximation of* M^m *along* γ if the two manifolds have common tangent space at every point of γ.

In dimension 2, the question of existence has been settled for a long time. A constructive proof, under suitable assumptions, is already present in Do Carmo's textbook [6, pp. 195–197]. It turns out the existence of a flat approximation of M^2 along γ implies the existence of a rolling, in Nomizu's sense, of M^2 on the tangent space $T_{\gamma(0)}M^2$ along the given curve—see [11,13]. More recently, Izumiya and Otani have shown uniqueness [8, Corollary 6.2].

In this paper, we extend the result in [6] to any curve in M^m. More precisely, we shall present a constructive proof of the following

Theorem 1.1. *Let $\gamma : I \to M^m$ be a smooth curve in a hypersurface M^m in $\mathbb{R}^{m+1}$. If the curve is never parallel to an asymptotic direction of M^m, then there exists a flat approximation H of M^m along γ. Such hypersurface is unique in the following sense: if H_1 and H_2 are two flat approximations of M^m along γ, then they coincide on an open set containing $\gamma(I)$.*

I. Markina: Department of Mathematics, University of Bergen, 5020 Bergen, Norway. e-mail: irina.markina@uib.no

M. Raffaelli ( ): DTU Compute, Technical University of Denmark, 2800 Kongens Lyngby, Denmark. e-mail: matraf@dtu.dk

Mathematics Subject Classification: 53A07 (primary) · 53B20 (secondary)

https://doi.org/10.1007/s00229-018-1072-6

Published online: 10 October 2018

The strategy to prove this result involves looking for $(m-1)$-tuples of linearly independent vector fields $(X_1, \dots, X_{m-1})$ along γ satisfying $\dot{\gamma}(t) \notin \operatorname{span}(X_j(t))_{j=1}^{m-1}$ for all t and having zero *normal* derivative (normal projection of Euclidean covariant derivative). Indeed, such conditions guarantee the image of the map $\gamma + \operatorname{span}(X_j)_{j=1}^{m-1}$ be a flat hypersurface of $\mathbb{R}^{m+1}$ in a sufficiently small neighbourhood of γ. The main difficulty resides in getting around the many-to-one correspondence between tuples of vector fields and rank-$(m-1)$ distributions along γ.

It is worth pointing out that the solution depends on the original hypersurface M^m only through its distribution of tangent planes along γ. Thus, when $m=2$, our problem is nothing but the classical Björling's problem—to find all minimal surfaces passing through a given curve with prescribed tangent planes—addressed to a different class of surfaces. In this respect, the present work joins several other recent studies aimed at solving Björling-type questions, see [2,3] and references therein.

The paper is organised as follows. The next two sections present some preliminaries, mostly for the sake of introducing relevant notation and terminology. In Sect. 4 we derive a simple condition for discerning when a parametrised ruled hypersurface has a flat metric. Such condition is then used in Sect. 5 to prove the main theorem. Finally, in Sect. 6 we give some general remarks about the construction of the approximation.

As a notational remark, beware that in this article we always use Einstein summation convention: every time the same index appears twice in any monomial expression, once as an upper index and once as a lower index, summation over all possible values of that index is understood.

Note. During the revision process, we have discovered that the authors in [7] provide an alternative viewpoint on the problem treated in this paper. We have also found out that a different method for constructing the solution could be deduced from [5, Section 2].

2. Vector cross products

Let V be an n-dimensional, real vector space equipped with a positive definite inner product $\langle\cdot,\cdot\rangle$. In the following, V^k will indicate the k-th Cartesian power of V, and $L^k(V)$ the set of all multilinear maps from V^k to V. Note that, under pointwise addition and scalar multiplication, $L^k(V)$ is a *finite dimensional* vector space, in that it is naturally isomorphic to the space $T^{(1,k)}(V)$ of tensors on V of type $(1,k)$—see for example [10, Lemma 2.1]. Thus, $\dim L^k(V) = n^{k+1}$.

A *k-fold vector cross product on V*, $1 \le k \le n$, is an element of $L^k(V)$—i.e., a multilinear map $X\colon V^k \to V$—satisfying the following two axioms:

$$\langle X(v_1, \dots, v_k), v_i\rangle = 0\,, \quad 1 \le i \le k.$$
$$\langle X(v_1, \dots, v_k), X(v_1, \dots, v_k)\rangle = \det(\langle v_i, v_j\rangle).$$

We emphasize that the second axiom implies any such X being alternating.

In particular, in the case V carries an orientation $\mathcal{O}$, we say that an $(n-1)$-fold vector cross product X is *positively oriented* if the following condition holds for all $(n-1)$-tuples of linearly independent vectors $v_1, \dots, v_{n-1}$:

$$(v_1, \dots, v_{n-1}, X(v_1, \dots, v_{n-1})) \in \mathcal{O}.$$

Analogously, a *negatively oriented* vector cross product satisfies the same relation with $-\mathcal{O}$ in place of $\mathcal{O}$.

In [4], Brown and Gray proved the following theorem:

Theorem 2.1. *Let V be an oriented finite dimensional inner product space, of dimension n. There exists a unique positively oriented $(n-1)$-fold vector cross product $X = \cdot \times \cdots \times \cdot$ on V. It is given by:*

$$v_1 \times \cdots \times v_{n-1} = \star(v_1 \wedge \cdots \wedge v_{n-1})$$

where $\star$ is the Hodge star operator on V.

We now turn our attention to manifolds. If M is a smooth Riemannian manifold of dimension m, let L^kTM be the disjoint union of all the vector spaces $L^k(T_pM)$:

$$L^kTM = \bigsqcup_{p\in M} L^k(T_pM).$$

Clearly, for $L^k(T_pM) \cong T^{(1,k)}(T_pM)$, the set L^kTM has a canonical choice of topology and smooth structure turning it into a smooth vector bundle of rank m^{k+1} over M. We define a *k-fold vector cross product on M*, where $1 \leq k \leq m$, to be a smooth section X of L^kTM such that, for every point $p \in M$, the map X_p is a k-fold vector cross product on T_pM.

We thus have the following corollary of Theorem 2.1:

Corollary 2.2. *Let M be a smooth oriented m-dimensional Riemannian manifold. There exists a unique $(m-1)$-fold positively oriented vector cross product on M. It acts on $(m-1)$-tuples of vector fields $X_1, \dots, X_{m-1}$ on M by*

$$X_1 \times \cdots \times X_{m-1} = \star(X_1 \wedge \cdots \wedge X_{m-1}).$$

3. Frames along curves

In this section we review some basic facts about Euclidean submanifolds and orthonormal frames along curves.

Let us start with some notation. If $m \geq 2$, let M be an m-dimensional embedded submanifold of $\mathbb{R}^d$, and $\gamma\colon I = [0, \alpha] \to M$ a smooth unit-speed curve in M. Throughout this paper, $\mathbb{R}^d$ will always be equipped with the standard Euclidean metric $\overline{g}$, typically indicated by a dot "$\cdot$", and standard orientation. Thus, there is a natural choice of Riemannian metric on M: the induced metric $\iota^*\overline{g}$, i.e., the pullback of $\overline{g}$ by the inclusion $\iota\colon M \hookrightarrow \mathbb{R}^d$.

Working with submanifolds, it is customary to identify each tangent space T_pM with its image under the differential of ι. In so doing, the ambient tangent space $T_p\mathbb{R}^d$ splits as the orthogonal direct sum $T_pM \oplus N_pM$, where N_pM is the normal space of M at p. Thus, the set $\mathfrak{X}(M)$ of tangent vector fields *on* M becomes a proper subset of the set of vector fields *along* M, which we denote by $\overline{\mathfrak{X}}(M)$. If $X \in \mathfrak{X}(M)$ and $\Upsilon \in \overline{\mathfrak{X}}(M)$,

$$\overline{\nabla}_X \Upsilon = (\overline{\nabla}_X \Upsilon)^\top + (\overline{\nabla}_X \Upsilon)^\perp,$$

where $\overline{\nabla}$ is the Euclidean connection, $\top$ and $\perp$ are the orthogonal projections onto the tangent and normal bundle of M, and where the vector fields X and Υ are extended arbitrarily to $\mathbb{R}^d$. It turns out that the map $\mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ defined by

$$(X, Y) \mapsto (\overline{\nabla}_X Y)^\top$$

is a linear connection on M, called the tangential connection. In fact, it is no other than the (intrinsic) Levi-Civita connection ∇ of $(M, \iota^*\overline{g})$.

Similarly, indicating by $\mathfrak{X}(M)^\perp$ the set of normal vector fields along M, we define the normal connection on M as the map $\mathfrak{X}(M) \times \mathfrak{X}(M)^\perp \to \mathfrak{X}(M)^\perp$ given by

$$(X, N) \mapsto (\overline{\nabla}_X N)^\perp.$$

Let us recall that an orthonormal frame along γ is an m-tuple of smooth vector fields $(E_i)_{i=1}^m$ along γ such that $(E_i(t))_{i=1}^m$ is an orthonormal basis of $T_{\gamma(t)}M$ for all t; the frame $(E_i)_{i=1}^m$ is γ-adapted when $E_1 = \dot{\gamma}$. In particular, an orthonormal frame $(W_1, \dots, W_d)$ along a curve $\iota \circ \gamma$ in $\mathbb{R}^d$ is said to be *M-adapted* if $(W_i)_{i=1}^m$ spans the ambient tangent bundle over γ.

In the remainder of this section, we assume that M has codimension one in $\mathbb{R}^d$, i.e., that $d = m + 1$. Under such hypothesis, given any orthonormal frame $(E_i)_{i=1}^m$ along γ, we can construct an associated M-adapted orthonormal frame along $\iota \circ \gamma$ as follows. For $k = 1, \dots, m$, let $W_k = E_k$; then, for $k = m + 1$,

$$W_{m+1} = E_1 \times \cdots \times E_m,$$

so that $(W_1, \dots, W_{m+1})$ is the unique extension of $(E_i(t))_{i=1}^m$ to a positively oriented, orthonormal frame along $\iota \circ \gamma$.

Assume $(E_i)_{i=1}^m$ be γ-adapted. Denoting by D_t and $\overline{D}_t$ the covariant derivative operators determined by ∇ and $\overline{\nabla}$, respectively, we may write

$$\overline{D}_t E_i = D_t E_i + \tau_i W_{m+1}, \tag{1}$$

for some smooth function $\tau_i \colon I \to \mathbb{R}$. Clearly, should M be orientable, $\tau_i = \pm\, h(E_1, E_i)$, where h is the (scalar) second fundamental form of M determined by a choice of unit normal vector field. Moreover, it easily follows from orthonormality that

$$\overline{D}_t W_{m+1} = -\tau_1 E_1 - \cdots - \tau_m E_m.$$

4. Developable surfaces

The main purpose of this section is to generalize to higher dimensions the following well-known fact about ruled surfaces in $\mathbb{R}^3$—see for example [6, p. 194]:

Lemma 4.1. *Let I, J be open intervals. Further, let γ and X be curves $I \to \mathbb{R}^3$ such that the map $\sigma : I \times J \to \mathbb{R}^3$ given by*

$$\sigma(t, u) = \gamma(t) + uX(t)$$

is a smooth injective immersion. Then the Gauss curvature of $\sigma(I \times J)$ is zero precisely when γ and X satisfy $\dot{\gamma} \cdot \dot{X} \times X = 0$.

We shall begin with some definitions extending the classical notions of ruled and torse surface to arbitrary dimension, yet keeping the codimension fixed to 1. If $m \geq 2$, let H be a hypersurface in $\mathbb{R}^{m+1}$, as always smooth and embedded.

Definition 4.2. We say that H is a *ruled* surface if

(1) H is free of planar points, that is, there exists no point of H where the second fundamental form vanishes;
(2) there exists a *ruled structure on H*, that is, a foliation of H by open subsets of $(m-1)$-dimensional affine subspaces of $\mathbb{R}^{m+1}$, called *rulings*.

In particular, a ruled surface H is said to be a *torse surface* if, for every pair of points (p, q) on the same ruling, we have $T_pH = T_qH$, i.e., if all tangent spaces of H along a fixed ruling can be canonically identified with the same linear subspace of $\mathbb{R}^{m+1}$.

Remark 4.3. Although condition (1) in Definition 4.2 may seem overly restrictive, it gives any ruled surface H a desirable property. Namely, it ensures the existence of a *smooth* ruled parametrisation of H [15]. On the other hand, we will also need to work with the broader class of *generalised ruled hypersurfaces* obtained by relaxing such condition. It is well known that every generalised torse with planar points is made up of both standard torses and pieces of m-planes, always glued along a well-defined ruling.

Remember that any d-dimensional Riemannian manifold locally isometric to $\mathbb{R}^d$ is said to be *flat*. In particular, the classical term for hypersurfaces is *developable*, see [16, Section 1] for a detailed discussion on terminology. Remarkably, it turns out that

Theorem 4.4. ([16, Theorem 1]). *H is a torse surface if and only if it is free of planar points and, when equipped with the induced metric $\iota^*\overline{g}$, H becomes a flat Riemannian manifold.*

Corollary 4.5. *H is a generalised torse surface if and only if the induced metric on H is flat.*

Given a curve γ in $\mathbb{R}^{m+1}$, the following result is key for constructing ruled surfaces containing γ. Note that in its statement we use the canonical isomorphism between $\mathbb{R}^{m+1}$ and any of its tangent spaces to identify the vector fields $X_1, \dots, X_{m-1}$ along γ with curves in $\mathbb{R}^{m+1}$.

Lemma 4.6. *Let I be a closed interval. Let $\gamma\colon I \to \mathbb{R}^{m+1}$ be a smooth injective immersion. Let $(X_1, \dots, X_{m-1})$ be a smooth, linearly independent $(m-1)$-tuple of vector fields along γ such that $\dot\gamma(t) \times X_1(t) \times \dots \times X_{m-1}(t) \neq 0$ for all $t \in I$. Then there exists an open box V in $\mathbb{R}^{m-1}$ containing the origin such that the restriction to $I \times V$ of the map $\sigma : I \times \mathbb{R}^{m-1} \to \mathbb{R}^{m+1}$ defined by*

$$\sigma(t, u) = \gamma(t) + u^j X_j(t)$$

is a smooth embedding.

Proof. To show that σ restricts to an embedding, we first prove the existence of an open box V_1 such that $\sigma|_{I\times V_1}$ is a smooth immersion. Essentially, the statement will then follow by compactness of I.

Obviously, σ is immersive at (t, u) if and only if the length $\ell : I \times \mathbb{R}^{m-1} \to \mathbb{R}$ of the cross product of the partial derivatives of σ is non-zero at (t, u). Thus, define W_t to be the subset of $\{t\} \times \mathbb{R}^{m-1}$ where σ is immersive. It is an open subset in $\mathbb{R}^{m-1}$ because it is the inverse image of an open set under a continuous map, $W_t = \ell(t, \cdot)^{-1}(\mathbb{R}\backslash\{0\})$; it contains 0 by assumption. Thence, there exists an $\epsilon_t > 0$ such that the open ball $B(\epsilon_t, 0) \subset \mathbb{R}^{m-1}$ is completely contained in W_t. Letting $\epsilon_1 = \inf_{t\in I}(\epsilon_t)$, we can conclude that the restriction of σ to the box $I \times (-\epsilon_1/2, \epsilon_1/2)^{m-1}$ is a smooth immersion.

Now, being σ a smooth immersion on $I \times V_1$, it follows that every point of $I \times V_1$ has a neighbourhood on which σ is a smooth embedding. Let then W_t' be the subset of W_t where σ is an embedding. It is open in $\mathbb{R}^{m-1}$, and it contains the origin because γ is a smooth injective immersion of a compact manifold. From here we may proceed as before. □

Thus, for suitably chosen $V \subset \mathbb{R}^{m-1}$, we have verified that $H_\sigma = \operatorname{Im}\sigma|_{I\times V}$ is a hypersurface in $\mathbb{R}^{m+1}$, and $\mathcal{F}_\sigma = \{\sigma(t, V)\}_{t\in I}$ a ruled structure on it. Under such hypothesis, let us assume H_σ is orientable (this we can do, possibly limiting the analysis to an open subset). Then, we may pick out a smooth unit normal vector field N along H_σ by means of the m-fold cross product on $\mathbb{R}^{m+1}$, as follows. Letting

$$Z = \frac{\partial\sigma}{\partial t} \times \frac{\partial\sigma}{\partial u^1} \times \dots \times \frac{\partial\sigma}{\partial u^{m-1}}, \tag{2}$$

define $\widehat{N} = Z|Z|^{-1}$, and so $N = \widehat{N} \circ \sigma^{-1}$. In this situation, assuming there are no planar points, H_σ being a torse surface is equivalent to N being constant along each of the rulings. Thus, indicating with $\overline{\nabla}$ the Euclidean connection on $\mathbb{R}^{m+1}$, $(H_\sigma, \iota^*\overline{g})$ is flat if and only if, for all vector fields X tangent to $\mathcal{F}_\sigma$ on H_σ:

$$\overline{\nabla}_X N = 0. \tag{3}$$

In fact, by linearity—and writing ∂_j as a shorthand for $\frac{\partial}{\partial u^j}$—it suffices that (3) holds for the vector fields $\sigma_*(\partial_1), \dots, \sigma_*(\partial_{m-1})$ spanning the distribution corresponding to $\mathcal{F}_\sigma$. We may thereby express the developability condition for $(H_\sigma, \iota^*\overline{g})$ simply as

$$\partial_1 \widehat{N} = \cdots = \partial_{m-1} \widehat{N} = 0, \tag{4}$$

where we understand ∂_j as acting on the coordinate functions $\widehat{N}^1, \ldots, \widehat{N}^{m+1}$ of $\widehat{N}$ in the standard coordinate frame of $T\mathbb{R}^{m+1}$.

The next lemma finally translates (4) into $m-1$ conditions involving the vector fields $X_1, \ldots, X_{m-1}$ along γ, and represents the sought generalization of Lemma 4.1. It says that $\iota^*\overline{g}$ is a flat Riemannian metric precisely when $\overline{D}_t X_j = D_t X_j$ for every j, or equivalently when each of the normal projections $(\overline{D}_t X_1)^\perp, \ldots, (\overline{D}_t X_{m-1})^\perp$ vanishes identically.

Lemma 4.7. *Assume $\sigma|_{I\times V}$ is a smooth embedding. The hypersurface H_σ is a generalised torse surface if and only if the following equations hold:*

$$\begin{aligned} \dot{\gamma} \cdot \partial_1 Z \equiv \dot{\gamma} \cdot \overline{D}_t X_1 \times X_1 \times \cdots \times X_{m-1} &= 0 \\ &\vdots \\ \dot{\gamma} \cdot \partial_{m-1} Z \equiv \dot{\gamma} \cdot \overline{D}_t X_{m-1} \times X_1 \times \cdots \times X_{m-1} &= 0 \end{aligned} \tag{5}$$

Proof. Computing the partial derivatives of σ and substituting them into the expression (2) for Z, we get:

$$Z(t,u) = \{\dot{\gamma}(t) + u^i \overline{D}_t X_i(t)\} \times X_1(t) \times \cdots \times X_{m-1}(t),$$

from which the identity $\partial_j Z \equiv \overline{D}_t X_j \times X_1 \times \cdots \times X_{m-1}$ clearly follows. Thus, we need to prove that $\partial_1 \widehat{N} = \cdots = \partial_{m-1}\widehat{N} = 0$ if and only if $\partial_1 Z \cdot \dot{\gamma} = \cdots = \partial_{m-1} Z \cdot \dot{\gamma} = 0$. In fact, for $\partial_j Z$ is orthogonal to $X_1, \ldots, X_{m-1}$, it is enough to check that $\partial_1 \widehat{N} = \cdots = \partial_{m-1}\widehat{N} = 0$ if and only if $(\partial_1 Z)^\top = \cdots = (\partial_{m-1} Z)^\top = 0$. First, assume $\partial_j \widehat{N} = 0$. Since $\widehat{N} = Z|Z|^{-1}$, it follows by linearity of the tangential projection that

$$|Z|(\partial_j Z)^\top - Z^\top \partial_j |Z| = 0,$$

which is true exactly when $(\partial_j Z)^\top = 0$, as desired. To verify the converse, note that $(\partial_j N)^\perp = 0$ because N has unit length. Thus, again by linearity of $\top$,

$$\partial_j \widehat{N} = \frac{(\partial_j Z)^\top |Z| - Z^\top \partial_j |Z|}{|Z|^2}.$$

Since $Z^\top = 0$, the claim follows. □

5. Proof of the main result

Here we prove our main result, stated in Theorem 1.1 in the Introduction. The proof is constructive and is based on the fact that an Euclidean hypersurface without planar points has a flat induced metric precisely when it is a torse surface (Theorem 4.4). Let M be a hypersurface in $\mathbb{R}^{m+1}$ and γ a smooth curve in M, as defined at the beginning of Sect. 3. Denoting by $\mathfrak{X}(\gamma)$ the set of smooth, non-vanishing vector fields along γ, define an equivalence relation on the n-th Cartesian power $\mathfrak{X}(\gamma)^n$ of $\mathfrak{X}(\gamma)$ by the following rule:

$$\{(X_1, \dots, X_n) \sim (Y_1, \dots, Y_n)\} \Leftrightarrow \{\mathrm{span}(X_1, \dots, X_n) = \mathrm{span}(Y_1, \dots, Y_n)\}.$$

Let us indicate an element of the quotient $\mathfrak{X}(\gamma)^n/\sim$, that is, an element of $\mathfrak{X}(\gamma)^n$ up to equivalence, by $[X_1, \dots, X_n]$. We wish to find $[X_1, \dots, X_{m-1}]$ such that, for every $t \in I$ and integer j with $1 \le j \le m-1$, both the conditions

$$\dot{\gamma} \cdot \overline{D}_t X_j \times X_1 \times \cdots \times X_{m-1} = 0 \tag{6}$$

$$\dot{\gamma}(t) \times X_1(t) \times \cdots \times X_{m-1}(t) \neq 0 \tag{7}$$

are satisfied.

Once and for all, let us choose a γ*-adapted* orthonormal frame $(E_1, \dots, E_m)$ along γ. The first step is to rewrite (6) as an equation involving the $m(m-1)$ coordinate functions X_j^i of $X_1, \dots, X_{m-1}$ with respect to $(E_1, \dots, E_m)$. Differentiating covariantly $X_j = X_j^i E_i$ and substituting, we obtain

$$E_1 \cdot \left(\overline{D}_t X_j^i E_i + X_j^i \overline{D}_t E_i\right) \times X_1^i E_i \times \cdots \times X_{m-1}^i E_i = 0, \tag{8}$$

whereas, from (1),

$$\begin{aligned}\sum_{i=1}^m \overline{D}_t E_i &= \sum_{i=1}^m D_t E_i + E_{m+1} \sum_{i=1}^m \tau_i \\ &= \sum_{i=1}^m \{(D_t E_i \cdot E_1) E_1 + \cdots + (D_t E_i \cdot E_m) E_m\} + E_{m+1} \sum_{i=1}^m \tau_i.\end{aligned}$$

Now, given any ordered m-tuple $(i_1, \dots, i_m)$ of integers with $1 \le i_1 \le m+1$ and $1 \le i_k \le m$ for $k = 2, \dots, m$, a necessary condition for the m-fold cross product $E_{i_1} \times \cdots \times E_{i_m}$ to give either E_1 or $-E_1$ is that $i_1 = m+1$ and $i_k \neq 1$. It follows that (8) is equivalent to

$$E_1 \cdot X_j^i \tau_i E_{m+1} \times \left(X_1^2 E_2 + \cdots + X_1^m E_m\right) \times \cdots \times \left(X_{m-1}^2 E_2 + \cdots + X_{m-1}^m E_m\right) = 0. \tag{9}$$

In fact, $E_{i_1} \times \cdots \times E_{i_m} = \pm E_1$ if and only if $i_1 = m+1$ and the $(m-1)$-tuple $(i_2, \dots, i_m)$ is a permutation of $(2, \dots, m)$. In particular, if it is an *even* permutation, then the basis $(E_{m+1}, E_{i_2}, \dots, E_{i_m}, E_1)$ is *negatively* oriented, for transposing E_{m+1} and E_1 must give a positive basis, and so $E_{i_1} \times \cdots \times E_{i_m} = -E_1$. Thence, denoting by S_m^2 the group of permutations σ of $(2, \dots, m)$, we may write (9) simply as

$$-X_j^i \tau_i \sum_{\sigma \in S_m^2} \mathrm{sgn}(\sigma) X_1^{\sigma(2)} \cdots X_{m-1}^{\sigma(m)} = 0.$$

On the other hand, a similar computation would reveal that condition (7) is satisfied for every t if and only if the summation term above (the term independent of j) never vanishes. We may thereby conclude that, under the assumption of (7) being true, condition (6) is equivalent to $X_j^i \tau_i = 0$.

Next, consider the set $\mathcal{Z} \subset \mathfrak{X}(\gamma)$ of smooth vector fields Z along γ such that $Z^1(t) = Z \cdot E_1(t) \neq 0$ for every t. We establish a bijection between its quotient $\mathcal{Z}/\sim$ by $\sim$ and the subset of $\mathfrak{X}(\gamma)^{m-1}/\sim$ where (7) holds. For every j, let

$$X_j(Z) = Z \times E_2 \times \cdots \times \widetilde{E}_{m-j+1} \times \cdots \times E_m, \tag{10}$$

where the tilde indicates that E_{m-j+1} is omitted, so that the cross product is $(m-1)$-fold. For example, when $j = 1$, we omit the last vector field E_m; when $j = 2$ the second to last, and so on, until dropping E_2 for $j = m - 1$. Linear independence of $E_1, X_1(Z), \ldots, X_{m-1}(Z)$ is easily seen, as by definition Z is never in the span of $E_2, \ldots, E_m$. Since the normal projection $Z \mapsto Z^\perp$ induces a bijection between $\mathcal{Z}/\sim$ and the set of smooth $(m-1)$-distributions along γ nowhere parallel to E_1, it follows that the map $[Z] \mapsto [X_1(Z), \ldots, X_{m-1}(Z)]$ between classes of equivalence is indeed a valid parametrisation of the solution set of (7).

We then compute the coordinates of the cross product in (10) with respect to the frame $(E_1, \ldots, E_m)$. Substituting $Z = Z^i E_i$, all but the terms $Z^1 E_1$ and $Z^{m-j+1} E_{m-j+1}$ will not give any contribution. In particular, $E_1 \times \cdots \times \widetilde{E}_{m-j+1} \times \cdots \times E_m = \pm E_{m-j+1}$ depending on whether $(E_1, \ldots, \widetilde{E}_{m-j+1}, \ldots, E_m, E_{m-j+1})$ is positively or negatively oriented. Since the corresponding permutation of $(1, \ldots, m)$ has sign $(-1)^{j-1}$, we conclude that $X_j^{m-j+1}(Z) = (-1)^{j-1} Z^1$. An analogous argument would show that $X_j^1(Z) = (-1)^j Z^{m-j+1}$.

Summing up, solving the original problem on $\mathfrak{X}(\gamma)^{m-1}/\sim$ essentially amounts to finding $[Z] \in \mathcal{Z}/\sim$ such that $X_j^i(Z)\tau_i = 0$ for every j. Moreover, by the previous computation,

$$X_j^i(Z)\tau_i = (-1)^j Z^{m-j+1}\tau_1 + (-1)^{j-1} Z^1 \tau_{m-j+1}.$$

Thus, denoting again by $\sim$ the equivalence relation on $C^\infty(I)^m = C^\infty(I; \mathbb{R}^m)$ naturally induced from the one on $\mathfrak{X}(\gamma)$, we need to look for $(Z^1, \ldots, Z^m)$, up to equivalence, satisfying the following system of $m - 1$ linear equations on $C^\infty(I; \mathbb{R}_{\neq 0}) \times C^\infty(I)^{m-1}$:

$$\begin{aligned} Z^m \tau_1 - Z^1 \tau_m &= 0 \\ Z^{m-1} \tau_1 - Z^1 \tau_{m-1} &= 0 \\ &\vdots \\ Z^3 \tau_1 - Z^1 \tau_3 &= 0 \\ Z^2 \tau_1 - Z^1 \tau_2 &= 0. \end{aligned} \tag{11}$$

Assume γ be not parallel to any asymptotic direction, so that $\tau_1(t) \neq 0$ for all t. Then, for any given Z^1 (remember Z^1 is non-vanishing by definition), the system has solution

$$\frac{Z^1}{\tau_1}(\tau_1, \ldots, \tau_m).$$

However, it is easy to see that all solutions are in one and the same equivalence class. Indeed, if f and g are two distinct values of Z^1, then

$$\frac{\tau_i}{\tau_1} f = \frac{f}{g} \frac{\tau_i}{\tau_1} g.$$

In particular, letting $Z^1 = \tau_1$, we obtain $Z^i = \tau_i$ for every $i = 1, \ldots, m$, and the solution of the original problem on $\mathfrak{X}(\gamma)^{m-1}/\sim$ is given by

$$\begin{aligned}
X_1 &= -\tau_m E_1 + \tau_1 E_m \\
X_2 &= \tau_{m-1} E_1 - \tau_1 E_{m-1} \\
&\vdots \\
X_{m-2} &= (-1)^{m-2} \tau_3 E_1 + (-1)^{m-3} \tau_1 E_3 \\
X_{m-1} &= (-1)^{m-1} \tau_2 E_1 + (-1)^{m-2} \tau_1 E_2.
\end{aligned}$$

As for uniqueness, in view of Remark 4.3, it is sufficient to show that the condition $\tau_1(t) \neq 0$ for all t implies any flat approximation H of M^m along γ be free of planar points, i.e., be a torse surface. To see this, let $N \equiv N \circ \sigma$ be a smooth unit normal vector field along H defined in a neighbourhood $I_t \times V$ of $(t, 0)$. Since by construction $N(\cdot, 0) = \pm E_{m+1}$ and $N(\cdot, u) = N(\cdot, 0)$ for all $u \in V$, it is clear that $\overline{D}_t N(\cdot, 0)(t) \neq 0$ if $\tau_1(t) \neq 0$. Hence the second fundamental form of H cannot vanish if τ_1 is always non-zero.

6. Construction of an adapted frame

As seen in the last section, the construction of the flat approximation of M along γ requires choosing some γ-adapted orthonormal frame $(E_i)_{i=1}^m$ along γ. We emphasize that such a choice is completely arbitrary. If the curve in question satisfies some (rather strong) conditions on its derivatives, then a natural generalization of the classical Frenet–Serret frame is available. The reader may find details on this construction in [9, 14]. Here we briefly review an alternative approach, one that does not require any initial assumption on the curve. Such approach is due to Bishop [1].

First of all, since the problem is local, we are free to assume that γ is a smooth embedding. Thus, for any point $p \in S = \gamma(I)$, there exist slice coordinates $(x_1, \ldots, x_m)$ in a neighbourhood U of p. It follows that $(\partial_1|_p, \ldots, \partial_m|_p)$ is a γ-adapted basis of $T_p M$, i.e., it satisfies $T_p S = \operatorname{span} \partial_1|_p$ and $N_p S = \operatorname{span}(\partial_2|_p, \ldots, \partial_m|_p)$. By applying the Gram–Schmidt process to these vectors, one obtains an orthonormal basis (n_j) of $N_p S$. Although this basis is by no means canonical, the normal connection $\nabla^\perp$ of S provides an obvious means for extending it to a frame for the normal bundle of S: for each j, let Υ_j be the unique normal parallel vector field along γ such that $\Upsilon_j|_p = n_j$—see [12, p. 119]. Because normal parallel translation is an isometry, the frame $(\dot{\gamma}, \Upsilon_1, \ldots, \Upsilon_{m-1})$ is an orthonormal adapted frame along γ, as desired.

Acknowledgements We thank the referee for the careful reading and valuable suggestions. We also thank J. Bohr and S. Markvorsen for useful comments on earlier versions of the manuscript.

References

[1] Bishop, R.L.: There is more than one way to frame a curve. Am. Math. Mon. **82**, 246–251 (1975)
[2] Brander, D., Svensson, M.: The geometric Cauchy problem for surfaces with Lorentzian harmonic Gauss maps. J. Differ. Geom. **108**, 411–457 (2018)
[3] Brander, D., Wang, P.: On the Björling problem for Willmore surfaces. J. Differ. Geom. **93**, 37–66 (2013)
[4] Brown, R.B., Gray, A.: Vector cross products. Comment. Math. Helv. **42**, 222–236 (1967)
[5] Dajczer, M., Vlachos, T.: The infinitesimally bendable Euclidean hypersurfaces. Ann. Mat. Pura Appl. (4) **196**, 1961–1979 (2017)
[6] do Carmo, M.P.: Differential Geometry of Curves and Surfaces. Prentice-Hall, Englewood Cliffs (1976)
[7] Fisher, R.J., Laquer, H.T.: Hyperplane envelopes and the Clairaut equation. J. Geom. Anal. **20**, 609–650 (2010)
[8] Izumiya, S., Otani, S.: Flat approximations of surfaces along curves. Demonstr. Math. **48**, 217–241 (2015)
[9] Kühnel, W.: Differential Geometry: Curves, Surfaces, Manifolds. Student Mathematical Library 77, 3rd edn. American Mathematical Society, Providence (2015)
[10] Lee, J.M.: Riemannian Geometry: An Introduction to Curvature. Graduate Texts in Mathematics 176. Springer, New York (1997)
[11] Nomizu, K.: Kinematics and differential geometry of submanifolds. Tohoku Math. J. **30**, 623–637 (1978)
[12] O'Neill, B.: Semi-Riemannian Geometry with Applications to Relativity. Academic Press, New York (1983)
[13] Raffaelli, M., Bohr, J., Markvorsen, S.: Sculpturing surfaces with Cartan ribbons. In: Proceedings of Bridges 2016: Mathematics, Music, Art, Architecture, Education, Culture, pp. 457–460. Tessellations Publishing, Phoenix (2016)
[14] Spivak, M.: A Comprehensive Introduction to Differential Geometry, vol. 4, 3rd edn. Publish or Perish, Houston (1999)
[15] Ushakov, V.: Parameterisation of developable surfaces by asymptotic lines. Bull. Austral. Math. Soc. **54**, 411–421 (1996)
[16] Ushakov, V.: Developable surfaces in Euclidean space. J. Austral. Math. Soc. Ser. A **66**, 388–402 (1999)

THE GEOMETRIC CAUCHY PROBLEM FOR DEVELOPABLE SUBMANIFOLDS

MATTEO RAFFAELLI

Abstract. Given a smooth distribution $\mathscr{D}$ of m-dimensional planes along a smooth regular curve γ in $\mathbb{R}^{m+n}$, we consider the following problem: To find an m-dimensional developable submanifold of $\mathbb{R}^{m+n}$, that is, a ruled submanifold with constant tangent space along the rulings, such that its tangent bundle along γ coincides with $\mathscr{D}$. In particular, we give sufficient conditions for the local well-posedness of the problem, together with a parametric description of the solution.

Contents

1. Introduction and main result

Given a smooth $(m + n)$-manifold Q^{m+n} and some class $\mathcal{A}^m$ of m-dimensional embedded submanifolds of Q^{m+n}, we can formulate the *geometric Cauchy problem* for the class $\mathcal{A}^m$ as follows:

Problem 1.1. Let $\gamma\colon I \to Q^{m+n}$ be a smooth regular curve in Q^{m+n}, and let $\mathscr{D}$ denote a smooth distribution of rank m along γ, such that $\dot\gamma(t) \in \mathscr{D}_t$ for all $t \in I$. Find all members of $\mathcal{A}^m$ containing γ and whose tangent bundle along γ is precisely $\mathscr{D}$.

Remark 1.2. In case Q^{m+n} is a Riemannian manifold, let $\mathscr{D}_t^\perp$ be the orthogonal complement of $\mathscr{D}_t$ in the tangent space $T_{\gamma(t)}Q^{m+n}$. Then, problem 1.1 is of course equivalent to finding all members of $\mathcal{A}^m$ containing γ and whose *normal* bundle along γ coincides with the *orthogonal distribution* $\mathscr{D}^\perp = \bigcup_t \mathscr{D}_t^\perp$.

Date: December 1, 2018.
2010 *Mathematics Subject Classification.* Primary: 53A07; Secondary: 53B20.

This problem has its roots in the classical Björling problem for minimal surfaces in $\mathbb{R}^3$, and has recently been examined for several combinations of Q^{2+n} and $\mathcal{A}^2$, generally with $n = 1$, see e.g. [4, 6, 12, 3, 1].

In particular, in a joint work with Irina Markina [11], the author has studied the case of developable hypersurfaces in $\mathbb{R}^{m+1}$ (the case $m = 2$ is well-known, see e.g. [8, p. 195–197]). We showed that, so long as the normal curvature of γ is never vanishing, a solution exists, is locally unique, and may be constructed using a method alternative to the classical Gauss parametrization [7], see Appendix B.

The purpose of this note is twofold. On one hand, we aim to give a new and simpler proof of the main theorem in [11]. At the same time, we intend to generalize such result to the whole class of developable submanifolds of $\mathbb{R}^{m+n}$. These are precisely the ruled submanifolds having no planar point and whose induced metric is flat, see Theorem 3.4.

In order to state our main theorem, set $Q^{m+n} = \mathbb{R}^{m+n}$. Let $\pi^\top$ and $\pi^\perp$ denote the orthogonal projections onto $\mathscr{D}$ and $\mathscr{D}^\perp$, respectively. Let $\overline{D}_t$ be the Euclidean covariant derivative along γ.

Theorem 1.3. *Assume the function $\pi^\perp(\overline{D}_t\dot\gamma)$ is never zero. The geometric Cauchy problem for developable submanifolds of $\mathbb{R}^{m+n}$ has a solution if and only if the linear map $\rho_t\colon \mathscr{D}_t^\perp \to \mathscr{D}_t$ defined by $\nu \mapsto \pi^\top(\overline{D}_t\nu)$ has rank one for every $t \in I$. Moreover:*

- (i) *The solution is unique in the following sense: if M_1 and M_2 are two solutions of the Cauchy problem, then they coincide on an open set containing $\gamma(I)$.*
- (ii) *In such neighborhood, the unique solution M satisfies $M = \gamma + \mathscr{D} \cap (\operatorname{Im}\rho)^\perp$.*
- (iii) *When I is closed, M can be parametrized as follows. Let $(E_1, \dots, E_m)$ be a smooth orthonormal frame for $\mathscr{D}$ satisfying $E_1 = \dot\gamma$. Choose a smooth section N of $\mathscr{D}^\perp$ such that $\pi^\top(\overline{D}_tN)$ is non-vanishing. For any $j = 1, \dots, m-1$, let*
$$X_j = \big(\overline{D}_t E_{j+1} \cdot N\big)\, E_1 - \big(\overline{D}_t E_1 \cdot N\big)\, E_{j+1}\,.$$
For sufficiently small $\varepsilon > 0$, let $\sigma\colon I \times (-\varepsilon, \varepsilon)^{m-1} \to \mathbb{R}^{m+n}$ be defined by
$$\sigma(t, u^1, \dots, u^{m-1}) = \gamma(t) + u^1 X_1(t) + \dots + u^{m-1} X_{m-1}(t)\,.$$
Then, $M = \operatorname{Im}\sigma$.

Remark 1.4. Whenever I is not closed, we may need to allow the parametrized solution to be defined on a subset of $\mathbb{R}^m$ different than a box. More precisely, if the function $\pi^\perp(\overline{D}_t\dot\gamma)$ has some zero in the closure of I, then we will need to have $\varepsilon = \varepsilon(t)$, with $\inf_{t\in I}\varepsilon(t) = 0$.

Remark 1.5. The existence condition may be equivalently stated as follows: the function $\pi^\perp(\overline{D}_t\dot\gamma)$ is never zero and there exists a smooth orthonormal frame $(N_1^*, \dots, N_n^*)$ for $\mathscr{D}^\perp$ such that $\pi^\top(\overline{D}_tN_k^*) = 0$ for all but one value of $k \in \{1, \dots, n\}$.

Note that, when $n = 1$, the non-vanishing of $\pi^\perp(\overline{D}_t\dot\gamma)(t)$ becomes a sufficient condition for the rank of ρ_t to be one, and we thus retrieve Theorem 1.1 in [11].

The paper is organized as follows. In Section 2 we review some background material. In Section 3 we derive a simple criterion for discerning when a parametrized ruled submanifold is developable. Such criterion, extending a well-known result of Yano [14], is of independent interest. In Section 4, using an approach based on Grassmannians, we prove Theorem 1.3. In Section 5 we present a sufficient condition for the solution to be a hypersurface in substantial codimension. Finally, in Section 6 we apply our main result to the problem of approximating – locally along a curve – a given submanifold by a developable one. There follow two appendixes: the first indicates a different method for proving Theorem 1.3. For the sake of completeness, in Appendix B we review a simpler parametrization, available in the case where $n = 1$.

Notation. In this paper, the integers i, j, k satisfy $i \in \{1, \dots, m\}$, $j \in \{1, \dots, m-1\}$ and $k \in \{1, \dots, n\}$, where $m \geq 2$ and $n \geq 1$. Note that we always use Einstein summation convention.

2. Preliminaries

2.1. **The wedge product.** Let V be an d-dimensional real vector space and let V^* be its dual space. A *tensor of type* (l, r) on V is a multilinear map

$$F\colon\ (V^*)^l \times V^r \to \mathbb{R}\,.$$

The set of all such tensors – which is of course a vector space under pointwise addition and scalar multiplication – we denote by $T^{(l,r)}(V)$.

Recall that a multilinear map is called *alternating* if its value changes sign whenever two arguments are interchanged. In particular, an alternating tensor of type $(0, r)$ is called a r-*covector* on V, whereas one of type $(l, 0)$ an l-*vector* on V. As usual, the sets of all l-vectors is denoted by $\Lambda^l(V)$, and we let $\Lambda(V) = \Lambda^1(V) \oplus \cdots \oplus \Lambda^d(V)$.

Given $\lambda \in \Lambda^r(V)$ and $\theta \in \Lambda^l(V)$, we define the *wedge product* $\lambda \wedge \theta$ to be the following $(r + l)$-vector:

$$\lambda \wedge \theta = \frac{(r+l)!}{r!\,l!}\,\mathrm{Alt}(\lambda \otimes \theta)\,,$$

where Alt denotes alternation [10, p. 351] and $\otimes$ is the ordinary tensor product.

Being bilinear, the wedge product turns the vector space $\Lambda(V)$ into an (associative, anticommutative graded) algebra, called the *exterior algebra* of V.

Given $v_1, \dots, v_l \in V$ and $\eta^1, \dots, \eta^l \in V^*$, it is easy to see that

$$v_1 \wedge \cdots \wedge v_l(\eta^1, \dots, \eta^l) = \det(\eta^\alpha(v_\beta))\,.$$

Moreover, we have the following lemma:

Lemma 2.1 ([10, Exercise 14-4])**.** *An l-tuple $(v_1, \dots, v_l)$ of elements of V is linearly dependent if and only if $v_1 \wedge \cdots \wedge v_l = 0$. Moreover, two l-tuples $(v_1, \dots, v_l)$ and $(w_1, \dots, w_l)$ have the same span if and only if there exists a non-zero real number λ such that:*

$$v_1 \wedge \cdots \wedge v_l = \lambda\, w_1 \wedge \cdots \wedge w_l\,.$$

2.2. **Grassmannians.** The *Grassmannian* $G(l, V)$ is the set of all l-dimensional linear subspaces of V. Once a basis of V has been chosen, we may identify $G(l, V)$ with the quotient $\mathcal{A}^{l\times d}/\!\sim$, where $\mathcal{A}^{l\times d}$ denotes the set of real $l \times d$ matrices of rank l,

$$\mathcal{A}^{l\times d} = \{A \in \mathbb{R}^{l\times d} \mid \operatorname{rank} A = l\},$$

and $\sim$ the equivalence relation

$$A \sim B \;\Leftrightarrow\; \text{there is a matrix } g \in \mathrm{GL}(l, \mathbb{R}) \text{ such that } B = gA.$$

Note that $A \sim B$ if and only if A and B have the same row space.

One may show that $\mathcal{A}^{l\times d}/\!\sim$, with the quotient topology, is a compact topological manifold of dimension $l(d-l)$. In fact, it has a natural smooth structure:

Let π be the canonical projection $\mathcal{A}^{l\times d} \to \mathcal{A}^{l\times d}/\!\sim$. Let J be any strictly ascending multi-index $1 \le i_1 < \cdots < i_l \le d$ of length l. For $A \in \mathcal{A}^{l\times d}$, let A_J be the $l \times l$ submatrix of A consisting of the i_1th$, \ldots, i_l$th columns of A. Then define

$$V_J = \{A \in \mathcal{A}^{l\times d} \mid \det A_J \neq 0\},$$

and $\tilde{\phi}_J \colon V_J \to \mathbb{R}^{l\times(d-l)}$,

$$\tilde{\phi}_J(A) = (A_J^{-1}A)_{J'},$$

where $(\;)_{J'}$ denotes the $l \times (d-l)$ submatrix obtained from the complement J' of the multi-index J. Finally, let $U_J = V_J/\!\sim$, and ϕ_J such that $\hat{\phi}_J = \phi_J \circ \pi$. It is standard to prove that $\{(U_J, \phi_J)\}$ is a smooth atlas for $\mathcal{A}^{l\times d}/\!\sim$.

2.3. **Distributions along curves.** Let γ be a smooth regular curve $I \to \mathbb{R}^{m+n}$. Without loss of generality, we may assume γ be unit-speed. Recall that the *ambient tangent bundle* $T\mathbb{R}^{m+n}|_\gamma$ over γ is the smooth vector bundle over I defined as the disjoint union of the tangent spaces of $\mathbb{R}^{m+n}$ at all points of $\gamma(I)$:

$$T\mathbb{R}^{m+n}|_\gamma = \bigsqcup_{t\in I} T_{\gamma(t)}\mathbb{R}^{m+n}\,.$$

We define a *distribution of rank m along γ* to be a smooth rank-m subbundle of the ambient tangent bundle over γ.

Let $\mathscr{D}$ be a distribution of rank m along γ, such that $\dot{\gamma}(t) \in \mathscr{D}_t$ for all $t \in I$. The standard Euclidean metric $\overline{g}$ on $\mathbb{R}^{m+n}$ allows us to decompose $T\mathbb{R}^{m+n}|_\gamma$ into the orthogonal direct sum of $\mathscr{D}$ and its normal bundle $\mathscr{D}^\perp$. Indeed, letting $\mathscr{D}_t^\perp$ denote the orthogonal complement of $\mathscr{D}_t \subset T_{\gamma(t)}\mathbb{R}^{m+n}$ with respect to $\overline{g}$, define $\mathscr{D}^\perp = \bigcup_t \mathscr{D}_t^\perp$, and so

$$T\mathbb{R}^{m+n}|_\gamma = \mathscr{D} \oplus \mathscr{D}^\perp\,. \tag{1}$$

In this setting, if v is an element of $T\mathbb{R}^{m+n}|_\gamma$, the *tangential projection* is the map $\pi^\top \colon \mathbb{R}^{m+n}|_\gamma \to \mathscr{D}$ defined by

$$v \mapsto \pi^\top(v)\,,$$

where $\pi^\top(v)$ is the orthogonal projection of v onto $\mathscr{D}$. Likewise, denoting by $\pi^\perp(v)$ the orthogonal projection of v onto $\mathscr{D}^\perp$, the *normal projection* $\pi^\perp$ is the map $\mathbb{R}^{m+n}|_\gamma \to \mathscr{D}^\perp$ defined by

$$v \mapsto \pi^\perp(v)\,.$$

Let now $(E_1, \dots, E_m)$ be a smooth γ-adapted orthonormal frame for $\mathscr{D}$: this is just an m-tuple of smooth vector fields along γ, such that $E_1 = \dot\gamma$, and such that $(E_i(t))_{i=1}^m$ is an orthonormal basis of $\mathscr{D}_t$ for all t. Similarly, let $(N_1, \dots, N_n)$ be an orthonormal frame for $\mathscr{D}^\perp$. It follows that the $(m+n)$-tuple $(E_1, \dots, E_m, N_1, \dots, N_n)$ is an orthonormal frame along γ; it respects the direct sum decomposition (1). Thence, denoting by $\overline{D}_t$ the Euclidean covariant derivative along γ, i.e., the covariant derivative along γ determined by the Levi-Civita connection of $(\mathbb{R}^{m+n}, \overline{g})$, we may write:

$$\overline{D}_t E_i = \pi^\top(\overline{D}_t E_i) + \tau_i^1 N_1 + \cdots + \tau_i^n N_n\,. \tag{2}$$

Here $\tau_i^1, \dots, \tau_i^n$ are smooth functions $I \to \mathbb{R}$. In particular, indicating $\overline{g}$ by a dot, $\tau_i^k = \overline{D}_t E_i \cdot N_k$.

3. The developability condition

In this section we aim to generalize a well-known result about ruled surfaces in $\mathbb{R}^{2+n}$:

Lemma 3.1 ([14])**.** *Let I, J be intervals. Further, let γ and X be curves $I \to \mathbb{R}^{2+n}$ such that the map $\sigma\colon I \times J \to \mathbb{R}^{2+n}$ given by*

$$\sigma(t,u) = \gamma(t) + uX(t)$$

is a smooth embedding. Then the tangent space of σ is constant along each ruling precisely when γ and X satisfy $\dot\gamma \wedge \dot X \wedge X = 0$.

To begin with, we shall extend the classical notion of ruled surface to arbitrary dimension:

Definition 3.2. An m-dimensional embedded submanifold M^m of $\mathbb{R}^{m+n}$ is a *ruled* submanifold if

1. M is free of planar points, that is, there exists no point of M where the second fundamental form vanishes;
2. there exists a *ruled structure on M*, that is, a foliation of M by open subsets of $(m-1)$-dimensional affine subspaces of $\mathbb{R}^{m+n}$, called *rulings*.

Following [13], we now define developable submanifolds and give two alternative characterizations of them.

Definition 3.3. The *relative nullity index* of M^m at a point p is the dimension of the *nullity space* Δ of M at p, which is the kernel of the second fundamental form α of M at p:

$$\Delta = \{x \in T_pM \mid \alpha(x, \cdot) = 0\}.$$

We say that M is a *developable* submanifold if for all $p \in M$ the relative nullity index is equal to $m-1$.

Theorem 3.4. *Let M be an m-dimensional embedded submanifold of $\mathbb{R}^{m+n}$. Let $\iota\colon M \hookrightarrow \mathbb{R}^{m+n}$ denote inclusion, and let $\overline{g}$ be the standard Euclidean metric on $\mathbb{R}^{m+n}$. The following statements are equivalent:*

(1) M is developable;

(2) M is ruled and for every pair of points (p, q) on the same ruling we have $T_pM = T_qM$, i.e., all tangent spaces of M along a fixed ruling can be canonically identified with the same linear subspace of $\mathbb{R}^{m+n}$;
(3) M is ruled and the induced metric on M is flat, that is, the Riemannian manifold $(M, \iota^*\overline{g})$ is locally isometric to $(\mathbb{R}^m, \overline{g})$.

Note that, if $n = 1$, then the theorem still holds when the requirement "M is ruled" in the third statement is replaced by "M is free of planar points". In other words, any flat hypersurface without planar points is automatically ruled.

Given a curve γ in $\mathbb{R}^{m+n}$, the following result is key for constructing ruled submanifold containing γ.

Lemma 3.5. *Let $\gamma\colon I \to \mathbb{R}^{m+n}$ be a smooth injective immersion, with I closed. Let $(X_1, \ldots, X_{m-1})$ be a smooth $(m-1)$-tuple of vector fields along γ such that $\dot\gamma(t) \wedge X_1(t) \wedge \cdots \wedge X_{m-1}(t) \neq 0$ for all $t \in I$. Then there exists an open box V in $\mathbb{R}^{m-1}$ containing the origin such that the restriction to $I \times V$ of the map $\sigma : I \times \mathbb{R}^{m-1} \to \mathbb{R}^{m+n}$ defined by*

$$\sigma(t, u) = \gamma(t) + u^j X_j(t)$$

is a smooth embedding.

Proof. To show that σ restricts to an embedding, we first prove the existence of an open box V_1 such that $\sigma|_{I\times V_1}$ is a smooth immersion. Essentially, the statement will then follow by compactness of I.

Obviously, σ is immersive at (t, u) if and only if the length $\ell : I \times \mathbb{R}^{m-1} \to \mathbb{R}$ of the wedge product of the partial derivatives of σ is non-zero at (t, u). Thus, define W_t to be the subset of $\{t\} \times \mathbb{R}^{m-1}$ where σ is immersive. It is an open subset in $\mathbb{R}^{m-1}$ because it is the inverse image of an open set under a continuous map, $W_t = \ell(t, \cdot)^{-1}(\mathbb{R} \setminus \{0\})$; it contains 0 by assumption. Thence, there exists an $\epsilon_t > 0$ such that the open ball $B(\epsilon_t, 0) \subset \mathbb{R}^{m-1}$ is completely contained in W_t. Letting $\epsilon_1 = \inf_{t\in I}(\epsilon_t)$, we can conclude that the restriction of σ to the box $I \times (-\epsilon_1/2, \epsilon_1/2)^{m-1}$ is a smooth immersion.

Now, being σ a smooth immersion on $I \times V_1$, it follows that every point of $I \times V_1$ has a neighborhood on which σ is a smooth embedding. Let then W_t' be the subset of W_t where σ is an embedding. It is open in $\mathbb{R}^{m-1}$, and it contains the origin because γ is a smooth injective immersion of a compact manifold. From here we may proceed as before. □

Thus, for suitably chosen $V \subset \mathbb{R}^{m-1}$, we have verified that $\sigma|_{I\times V}$ is an m-dimensional embedded submanifold of $\mathbb{R}^{m+n}$, and $\{\sigma(t, V)\}_{t\in I}$ a ruled structure on it. Under such hypothesis, letting

$$Z = \frac{\partial\sigma}{\partial t} \wedge \frac{\partial\sigma}{\partial u^1} \wedge \cdots \wedge \frac{\partial\sigma}{\partial u^{m-1}}\,, \tag{3}$$

we may express the constancy of the tangent space along the coordinate vector field $\frac{\partial\sigma}{\partial u^j}(t, \cdot)$ as follows: for each value of $u^j \neq 0$ there exists a (non-zero) real number λ such that

$$\lambda Z(t, 0) = Z(t, 0, \ldots, 0, u^j, 0, \ldots, 0)\,.$$

The next lemma translates this condition into an equation involving the vector fields $X_1, \dots, X_{m-1}$ along γ. As an easy corollary we obtain the desired generalization of Lemma 3.1.

Lemma 3.6. *Assume $\sigma|_{I\times V}$ be a smooth embedding. The tangent space of $\sigma|_{I\times V}$ is constant along $\frac{\partial\sigma}{\partial u^j}$ if and only if the following equation holds*

$$\overline{D}_t X_j \wedge \dot\gamma \wedge X_1 \wedge \cdots \wedge X_{m-1} = 0\,. \tag{4}$$

Proof. Computing the partial derivatives of σ and substituting them into the expression (3) for Z, we obtain

$$Z(t,u) = \left\{\dot\gamma(t) + u^j \overline{D}_t X_j(t)\right\} \wedge X_1(t) \wedge \cdots \wedge X_{m-1}(t)\,.$$

Hence, we need to show that

$$\overline{D}_t X_j(t) \wedge \dot\gamma(t) \wedge X_1(t) \wedge \cdots \wedge X_{m-1}(t) = 0 \tag{5}$$

if and only if for each $u^j \neq 0$ there exists λ such that

$$(\lambda - 1)\dot\gamma(t) \wedge X_1(t) \wedge \cdots \wedge X_{m-1}(t) = u^j \overline{D}_t X_j(t) \wedge X_1(t) \wedge \cdots \wedge X_{m-1}(t)\,.$$

First, assume, for some $u^j \neq 0$, that such a λ exists. If $\lambda = 1$, then the m-vector on the right hand side is necessarily zero. Else, if $\lambda \neq 1$, then $(\dot\gamma(t), X_1(t), \dots, X_{m-1}(t))$ and $(\overline{D}_t X_j(t), X_1(t), \dots, X_{m-1}(t))$ have the same span. Either way, it is clear that (5) holds. Conversely, if the vectors $\overline{D}_t X_j(t), \dot\gamma(t), X_1(t), \dots, X_{m-1}$ are linearly dependent, then there exist real numbers $a_1, \dots, a_m$ such that

$$\overline{D}_t X_j(t) = a_1 X_1(t) + \cdots + a_{m-1} X_{m-1}(t) + a_m \dot\gamma(t)\,.$$

It follows that

$$u^j \overline{D}_t X_j(t) \wedge X_1(t) \wedge \cdots \wedge X_{m-1}(t) = u^j a_m \dot\gamma(t) \wedge X_1(t) \wedge \cdots \wedge X_{m-1}(t)\,,$$

and so for any $u^j \neq 0$ the desired λ satisfies $\lambda - 1 = u^j a_m$. □

Corollary 3.7. *Assume $\sigma|_{I\times V}$ be a smooth embedding. Then $\sigma|_{I\times V}$ is developable if and only if it is ruled (i.e., without planar points) and the following $m-1$ equations are fulfilled:*

$$\begin{aligned} \overline{D}_t X_1 \wedge \dot\gamma \wedge X_1 \wedge \cdots \wedge X_{m-1} &= 0\,, \\ &\vdots \\ \overline{D}_t X_{m-1} \wedge \dot\gamma \wedge X_1 \wedge \cdots \wedge X_{m-1} &= 0\,. \end{aligned}$$

As a final result of this section, we prove the following proposition, which will be useful in the proof of Theorem 1.3.

Proposition 3.8. *If X_j is tangent to $\mathscr{D}$ along γ – i.e., $X_j(t) \in \mathscr{D}_t$ for every $t \in I$ – then (4) is equivalent to*

$$X^i_j \tau^1_i = \cdots = X^i_j \tau^n_i = 0\,, \tag{6}$$

where X^i_j denotes the i-th coordinate function of X_j with respect to $(E_1, \dots, E_m)$, and where $\tau^1_i, \dots \tau^n_i$ are defined by (2).

Proof. Clearly – assuming X_j be tangent to $\mathscr{D}$ along γ – equation (4) holds if and only if $\overline{D}_t X_j \cdot N_k = 0$ for all $k \in \{1, \dots, n\}$. Differentiating $X_j = X_j^i E_i$ and substituting in $\overline{D}_t X_j \cdot N_k = 0$, we obtain

$$X_j^i \overline{D}_t E_i \cdot N_k = 0\,,$$

for the term $\dot{X}_j^i E_i \cdot N_k$ vanishes. Using (2), this is equivalent to

$$X_j^i \left(\tau_i^1 N_1 + \cdots + \tau_i^n N_n\right) \cdot N_k = 0\,,$$

again because $X_j^i \pi^\top (\overline{D}_t E_i) \cdot N_k = 0$. □

4. Proof of Theorem 1.3

We are now ready to prove Theorem 1.3 in the Introduction. Before treating the general case, let us for now assume $n = 1$. To simplify notation, in this section we often write τ_i as a shorthand for $\tau_i(t)$.

Let $(x_1, \dots, x_{m-1})$ be a linearly independent $(m-1)$-tuple of vectors in $\mathscr{D}_t = \operatorname{span}(e_i)_{i=1}^m$, where $e_i = E_i(t)$. Denoting by x_j^i the i-th coordinate of x_j with respect to the basis $(e_i)_{i=1}^m$, we may identify the tuple $(x_1, \dots, x_{m-1})$ with the matrix

$$X = \begin{pmatrix} x_1^1 & \dots & x_1^m \\ \vdots & \ddots & \vdots \\ x_{m-1}^1 & \dots & x_{m-1}^m \end{pmatrix} \in \mathcal{A}^{(m-1)\times m}.$$

The problem is thus to find $[X] = [x_1, \dots, x_{m-1}] \in \mathcal{A}^{(m-1)\times m}/\sim$ such that, for every $j \in \{1, \dots, m-1\}$, both the conditions

$$x_j^i \tau_i = 0 \tag{7}$$

$$e_1 \wedge x_1 \wedge \cdots \wedge x_{m-1} \neq 0 \tag{8}$$

are satisfied. (Since we are assuming $n = 1$, we write τ_i for τ_i^1.)

First, we shall examine (8). It is easy to see that (8) corresponds to the requirement that the $(m-1) \times (m-1)$ submatrix $X_{2\cdots m-1}$ of X obtained by removing the first column of X has full rank. In other words, we just need to look for $[X] \in U_{2\cdots m-1}$ such that, for every j, equation (7) holds.

Define a map $\psi_{2\cdots m-1} \colon \mathbb{R}^{m-1} \to V_{2\cdots m-1}$ by

$$z = (z^1, \dots, z^{m-1}) \mapsto \begin{pmatrix} z^1 & 1 & 0 & \dots & 0 \\ z^2 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ z^{m-1} & 0 & 0 & \dots & 1 \end{pmatrix}.$$

Since $\phi_{2\cdots m-1}^{-1} = \pi \circ \psi_{2\cdots m-1}$ is a parametrization of $U_{2\cdots m-1}$, the original problem in $[X]$ reduces to the uncoupled system of equations $\{\psi_{2\cdots m-1}(z)_j^i \tau_i = 0\}_j = \{z^j \tau_1 + \tau_{j+1} = 0\}_j$ on $\mathbb{R}^{m-1}$.

Assume $\tau_1 \neq 0$. Then $z^j = -\tau_{j+1}/\tau_1$. Since $[X] = [-\tau_1 X]$, it follows that the tuple $(\tau_{j+1} e_1 - \tau_1 e_{j+1})_j$ represents the unique solution of our problem.

Let us now consider the case where n is arbitrary: equation (7) turns into the system

$$x_j^i \tau_i^1 = \cdots = x_j^i \tau_i^n = 0\,. \tag{9}$$

Assume $\pi^\perp(\overline{D}_t E_1)(t) \neq 0$. It follows from (2) that there exists $s \in \{1, \dots, n\}$ such that $\tau_1^s \neq 0$. The s-th system $\{x_j^i \tau_i^s = 0\}_j$ admits therefore the unique solution $(\tau_{j+1}^s e_1 - \tau_1^s e_{j+1})_j$. Clearly, the solution satisfies the remaining $n-1$ systems in (9) if and only if, for each $k \neq s$, either $\tau_1^k = \cdots = \tau_m^k = 0$ or

$$\frac{\tau_2^k}{\tau_1^k} = \frac{\tau_2^s}{\tau_1^s}, \ \dots, \ \frac{\tau_m^k}{\tau_1^k} = \frac{\tau_m^s}{\tau_1^s}.$$

In other words, precisely when the rank of ρ_t is one.

5. Codimension reduction

Let M^m be a submanifold of a Riemannian manifold $\widetilde{M}^{m+n}$, and let N_pM be the normal space of M at p. For $\nu \in N_pM$, we indicate by A_ν the shape operator of M in direction ν.

Recall that:

Definition 5.1 ([2, p. 16]). M is said to be a *full submanifold* if it is not contained in any totally geodesic submanifold S of $\widetilde{M}$ with $\dim S < \dim \widetilde{M}$. If M is not full, one says that there is a *reduction of the codimension of* M.

A key result about codimension reduction was given by J. Erbacher in 1971:

Theorem 5.2 ([9]). *Assume $\widetilde{M}^{m+n}$ is of constant sectional curvature. If the first normal space of M is invariant under parallel translation with respect to the normal connection and is of constant dimension l, then M is not full. In particular, M is contained in an $(m+l)$-dimensional totally geodesic submanifold of $\widetilde{M}^{m+n}$.*

Recall that the *first normal space* of M at a point $p \in M$ is the linear subspace N_p^1M of N_pM spanned by the image of the second fundamental form at p. In other words, N_p^1M is the orthogonal complement in N_pM of the kernel of the linear map $N_pM \to \mathrm{End}(T_pM)$, $\nu \mapsto A_\nu$. If the dimension of N_p^1M is constant on M, then N^1M is a smooth subbundle of the normal bundle of M.

As a corollary of Theorem 1.3, we obtain:

Corollary 5.3 ([13, Theorem 3]). *The dimension of the first normal space at any point of a developable submanifold of $\mathbb{R}^{m+n}$ is one.*

Combining Corollary 5.3 and Theorem 5.2 (see also Remark 1.5), we have:

Proposition 5.4. *Assume the function $\pi^\perp(\overline{D}_t\dot\gamma)$ is never zero and there exists a smooth orthonormal frame $(N_1^*, \dots, N_n^*)$ for $\mathscr{D}^\perp$ such that $\pi^\top(\overline{D}_t N_k^*) = 0$ for all but one value s of $k \in \{1, \dots, n\}$. If $\pi^\perp(\overline{D}_t N_s^*) = 0$, then the solution of the associated geometric Cauchy problem for developable submanifolds of $\mathbb{R}^{m+n}$ is not full. In particular, it is a hypersurface in an $(m+1)$-dimensional affine subspace of $\mathbb{R}^{m+n}$.*

Proof. □

6. Application to approximations

Given a submanifold M^m of $\mathbb{R}^{m+n}$ and a smooth regular curve γ in M, we call any submanifold containing γ and having the same tangent bundle as M along γ a (*first-order*) *approximation of M along γ*.

The result below follows easily from our main theorem:

Theorem 6.1. *Let α be the second fundamental form of M. Suppose the curve γ is never parallel to an asymptotic direction of M, i.e., that $\alpha(\dot\gamma,\dot\gamma)$ never vanishes. Suppose further that the linear map $\alpha_t = \alpha(\dot\gamma(t),\cdot)$ has rank one for all $t \in I$. Then there exists a developable approximation of M along γ. Such approximation is locally unique, and may be constructed as presented in Theorem 1.3 (iii).*

Proof. With the notations of Theorem 1.3, set $\mathscr{D}_t = T_{\gamma(t)}M$ and assume that $\alpha_t \colon x \mapsto \pi^\perp \overline{D}_t x$ has rank one: we shall show that $\operatorname{rank}\rho_t = 1$.

Note that, if $x \in \mathscr{D}_t$ and $\nu \in \mathscr{D}_t^\perp$, then

$$\rho_t(\nu)\cdot x = \overline{D}_t\nu\cdot x = -\nu\cdot\overline{D}_t x = -\nu\cdot\alpha_t(x)\,.$$

This shows that ρ_t and α_t are negative adjoint with respect to the dot product, and so have the same rank. ☐

Appendix A. Alternative proof of Theorem 1.3

In this appendix we present a coordinate-free approach for proving the main part of Theorem 1.3, up to and including (ii). We also sketch an alternative method, adapted from [11, Section 5], for obtaining the parametrized solution given in (iii).

Let $(X_j)_{j=1}^{m-1}$ be a smooth, linearly independent $(m-1)$-tuple of vector fields – always tangent to $\mathscr{D}$ – along γ. By Corollary 3.7, we need to find $(X_j)_{j=1}^{m-1}$ such that, for any section Y of $\mathscr{D}^\perp$ and any $j = 1,\dots,m-1$,

$$X_j\cdot\overline{D}_tY \equiv X_j\cdot\pi^\top(\overline{D}_tY) \equiv X_j\cdot\rho(Y) = 0. \tag{10}$$

Hence, our problem amounts to finding $\Sigma = \operatorname{span}(X_j)_{j=1}^{m-1}$, satisfying $\dot\gamma(t) \notin \Sigma_t$ for every t, and such that

$$\Sigma \subset (\operatorname{Im}\rho)^\perp \cap \mathscr{D}. \tag{11}$$

Here by $(\operatorname{Im}\rho)^\perp$ we mean of course the distribution $(\operatorname{Im}\rho_t)^\perp_{t\in I}$, where the superscript $^\perp$ denotes orthogonal complement in the ambient tangent space.

Assume $\pi^\perp(\overline{D}_t\dot\gamma)(t) \neq 0$ for all $t \in I$. Then there exists a smooth section N of $\mathscr{D}^\perp$ such that $\dot\gamma\cdot\overline{D}_tN = \dot\gamma\cdot\pi^\top(\overline{D}_tN)$ never vanishes. It follows that, for any t, $\operatorname{rank}\rho_t \neq 0$ and $\dot\gamma(t) \notin (\operatorname{Im}\rho_t)^\perp$. Since the dimension of the intersection in (11) equals $m - \operatorname{rank}\rho_t$, it is clear that a solution Σ exists if and only if $\operatorname{rank}\rho = 1$, and that such solution is given by equality in (11).

As for (iii), pick an orientation on $\mathscr{D}$. Associated to such a choice (and the natural bundle metric) there is a well-defined Hodge star operator $\star$ on $\mathscr{D}$, which in turn defines a unique $(m-1)$-fold vector cross product on $\mathscr{D}$ – see [5, Section 3]. This product acts on tuples of vector fields $X_1,\dots,X_{m-1}$ *on* $\mathscr{D}$ by

$$X_1\times\cdots\times X_{m-1} = \star(X_1\wedge\cdots\wedge X_{m-1})\,.$$

Let N as above. For every $j \in \{1, \ldots, m-1\}$, let

$$X_j(N) = \overline{D}_t N \times E_2 \times \cdots \widehat{E_{j+1}} \cdots \times E_m \,, \tag{12}$$

where the hat indicates that E_{j+1} is omitted, so that the cross product is $(m-1)$-fold for every j. Since N is never in the span of $E_2, \ldots, E_m$, it follows that $(X_1(N), \ldots, X_{m-1}(N))$ is linearly independent, i.e., $\operatorname{span}(X_j(N))_{j=1}^{m-1} = \overline{D}_t N^\perp$. By computing the coordinates of the cross product in (12) with respect to the frame $(E_1, \ldots, E_m)$, the desired expression is easily obtained.

Appendix B. Alternative construction for $n = 1$

In the case where $n = 1$, let N be a continuous section of $\mathscr{D}^\perp$ such that $N \cdot N = 1$. This section is automatically smooth; it is unique up to a sign. Assuming existence, the solution of the Cauchy problem for developable hypersurfaces is given by the distribution $\overline{D}_t N^\perp \cap N^\perp$. If we identify, through parallel translation in $\mathbb{R}^{m+1}$, the vector field N with a curve in the unit sphere $\mathbb{S}^m$, we have:

$$\overline{D}_t N^\perp \cap N^\perp|_t \equiv \dot{N}(t)^\perp \cap T_{N(t)}\mathbb{S}^m \,.$$

Hence, we may alternatively parametrize the solution using any smooth frame for the normal space of $N\colon I \to \mathbb{S}^m$.

References

1. Juan A. Aledo, José A. Gálvez, and Pablo Mira, *A D'Alembert formula for flat surfaces in the 3-sphere*, J. Geom. Anal. **19** (2009), 211–232.
2. Jürgen Berndt, Sergio Console, and Carlos Enrique Olmos, *Submanifolds and holonomy*, second ed., Monographs and Research Notes in Mathematics, CRC Press, Boca Raton, 2016.
3. David Brander and Martin Svensson, *The geometric Cauchy problem for surfaces with Lorentzian harmonic Gauss maps*, J. Differential Geom. **93** (2013), 37–66.
4. David Brander and Peng Wang, *On the Björling problem for Willmore surfaces*, J. Differential Geom. **108** (2018), 411–457.
5. Robert B. Brown and Alfred Gray, *Vector cross products*, Comment. Math. Helv. **42** (1967), 222–236.
6. Adriana A. Cintra, Francesco Mercuri, and Irene I. Onnis, *The Björling problem for minimal surfaces in a Lorentzian three-dimensional Lie group*, Ann. Mat. Pura Appl. (4) **195** (2016), 95–110.
7. Marcos Dajczer and Detlef Gromoll, *Gauss parametrizations and rigidity aspects of submanifolds*, J. Differential Geom. **22** (1985), 1–12.
8. Manfredo P. do Carmo, *Differential geometry of curves and surfaces*, Prentice-Hall, Englewood Cliffs, 1976.
9. Joseph Erbacher, *Reduction of the codimension of an isometric immersion*, J. Differential Geom. **5** (1971), 333–340.
10. John M. Lee, *Introduction to smooth manifolds*, second ed., Graduate Texts in Mathematics, no. 218, Springer, New York, 2013.
11. Irina Markina and Matteo Raffaelli, *Flat approximations of hypersurfaces along curves*, Manuscripta Math., to appear.
12. Antonio Martínez and Francisco Milán, *The geometric Cauchy problem for the hyperbolic Hessian one equation*, Nonlinear Anal. **125** (2015), 323–333.
13. Vitaly Ushakov, *Developable surfaces in Euclidean space*, J. Austral. Math. Soc. Ser. A **66** (1999), 388–402.
14. Kentaro Yano, *On the torse-forming directions in Riemannian spaces*, Proc. Imp. Acad. Tokyo **20** (1944), 340–345.

DTU Compute, Technical University of Denmark, 2800 Kongens Lyngby, Denmark
E-mail address: matraf@dtu.dk